Carson Slater

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STA 5353

Homework 7

1) $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Compare CI's for μ assuming

a) σ^2 known.

A $1-\alpha$ CI for μ when σ^2 is known is

$$1-\alpha = P\left(U(\mu) < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < L(\mu)\right)$$

$$= P\left(-z_{\alpha/2} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right)$$

$$= P\left(\bar{X} - z_{\alpha/2}\,\sigma/\sqrt{n} < \mu < \bar{X} + z_{\alpha/2}\,\sigma/\sqrt{n}\right)$$

So the shortest $1-\alpha$ CI for μ is

$$\left(\bar{X} - z_{\alpha/2}\,\sigma/\sqrt{n},\ \bar{X} + z_{\alpha/2}\,\sigma/\sqrt{n}\right)$$

from a prior homework result. So the expected length is

$$E\left[\bar{X} + z_{\alpha/2}\,\sigma/\sqrt{n} - \left(\bar{X} - z_{\alpha/2}\,\sigma/\sqrt{n}\right)\right] = E\left[2z_{\alpha/2}\,\sigma/\sqrt{n}\right]$$

$$= 2z_{\alpha/2}\,\sigma/\sqrt{n}.$$

b) When σ^2 unknown, we estimate σ^2 with S^2.

A $1-\alpha$ CI for μ when σ^2 is known is

$$1-\alpha = P\left(U(\mu) < \frac{\bar{X}-\mu}{S/\sqrt{n}} < L(\mu)\right)$$

Pivotal quantity from in-class slides follows a t_{n-1}.

$$= P\left(-t_{\alpha/2, n-1} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < t_{\alpha/2, n-1}\right)$$

$$= P\left(\bar{X} - t_{\alpha/2, n-1} \, S/\sqrt{n} < \mu < \bar{X} + t_{\alpha/2, n-1} \, S/\sqrt{n}\right)$$

So the shortest $1-\alpha$ CI for μ is (σ^2 unknown)

$$\left(\bar{X} - t_{\alpha/2, n-1} \, S/\sqrt{n}, \ \bar{X} + t_{\alpha/2, n-1} \, S/\sqrt{n}\right).$$

So the expect length is

$$E\left[\bar{X} + t_{\alpha/2, n-1} \, S/\sqrt{n} - \left(\bar{X} - t_{\alpha/2, n-1} \, S/\sqrt{n}\right)\right] = E\left[2 t_{\alpha/2, n-1} \, S/\sqrt{n}\right]$$

$$= 2 t_{\alpha/2, n-1} \frac{E[S]}{\sqrt{n}},$$ where $E[S]$ is found to be:

$$E[S] = \sqrt{\frac{\sigma^2}{n-1}} \, E\left[\sqrt{Y}\right]$$ (such that $Y \sim \chi^2_{n-1}$, $Y = \frac{(n-1)S^2}{\sigma^2}$)

$$= \sqrt{\frac{\sigma^2}{n-1}} \int_0^\infty \sqrt{y} \, \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{n-1/2}} \, y^{\frac{n-1}{2}} e^{-y/2} \, dy$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \, \frac{\Gamma\left(\frac{n}{2}\right) 2^{n/2-1}}{\Gamma\left(\frac{n-1}{2}\right) 2^{n-1/2}} \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} \, y^{\frac{n}{2}-1} e^{-y/2} \, dy$$

$$= \sqrt{\frac{2\sigma^2}{n-1}} \, \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\Rightarrow E\left[\text{Length}(C(X))\right] = 2 t_{\alpha/2, n-1} \sqrt{\frac{2\sigma^2}{n(n-1)}} \, \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

As $n \to \infty$, they converge to be the same interval but the σ^2 known interval is smaller otherwise.

2) $X_1, \ldots, X_n$ are independent with pdfs

$$f_{X_i}(x|\theta) = \exp\{i\theta - x\} I_{[i\theta, \infty]}(x) = \exp\{-(x - \theta i)\} I_{[\theta i, \infty)}(x)$$

a) To find a sufficient statistic for θ, we can write the joint pdf of the sample:

$$f(\underline{x}|\theta) = \exp\{-\sum_{i=1}^{n}(x - \theta i)\} I_{[\theta, \infty)}\left(\underline{\frac{x_{(1)}}{i}}\right)$$

So then $f_{X_i}(\underline{x}|\theta) = g_{X_i}(\underline{x}|\theta) h_{X_i}(\underline{x})$, where

notice how I divided by i

$$g(\underline{x}|\theta) = e^{n\theta} I_{[\theta, \infty)}\left(\frac{x_{(1)}}{i}\right)$$

$$h(\underline{x}) = e^{-\sum_{j=1}^{n} x_j}.$$

So then $T(\underline{x}) = \frac{X_{(1)}}{i}$ is sufficient for θ by the Factorization Theorem.

b) Since $\frac{X_{(1)}}{i}$ does not depend on θ,

The pdf of $X_{(1)}$ is $f_{X(1)}(X_{(1)}) = ne^{-n(X_{(1)} - \theta i)} I_{[\theta i, \infty)}(X_{(1)})$. Now consider the transformation $Y = \frac{X_{(1)}}{i}$. Since θ is a location parameter the pivotal quantity will be

$$Z = Y - \frac{\theta}{i} = \frac{X_{(1)} - \theta}{i} \Rightarrow iZ = X_{(1)} - \theta.$$

So then $$\frac{\alpha}{2} = \int_0^a \underbrace{nie^{-niz}} dz = -e^{-niz}\Big|_0^a = \underbrace{1 - e^{-nia}}_{C_{a_0}}$$

$$\Rightarrow -nia = \log\left(1 - \frac{\alpha}{2}\right) \Rightarrow a = -\frac{1}{in}\log\left(1 - \frac{\alpha}{2}\right)$$

and

$$\frac{\alpha}{2} = \int_b^{\infty} ni e^{-nv} = -e^{-niz}\Big|_b^{\infty} = e^{-nib}$$

$$\Rightarrow -nib = \log\frac{\alpha}{2} \Rightarrow b = -\frac{1}{in}\log\left(\frac{\alpha}{2}\right)$$

Hence, the pivotal interval (by the pivot method) would be $\left\{\theta : Y + \frac{1}{in}\log\left(\frac{\alpha}{2}\right) < \theta < Y + \frac{1}{in}\log\left(1-\frac{\alpha}{2}\right)\right\}$. So the $1-\alpha$ confidence interval is

$$\left(Y + \frac{1}{in}\log\left(\frac{\alpha}{2}\right), Y + \frac{1}{in}\log\left(1-\frac{\alpha}{2}\right)\right).$$

* I understand the answer key has something different; I have reviewed my mistakes and I kept my original solution for the sake of my own records so I could learn from my mistakes, especially since this is for a completion grade.

3) Consider $X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. Consider the test $H_0: \lambda_0 = \lambda$ and $H_a: \lambda > \lambda_0$. Now finding the UMA, we must find the UMP level α test.

First, we have that the joint pdf of $\underline{X}$ is $f(\underline{x} \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} I_{\{0,1,2,\dots\}}(x_i)$

$= \dfrac{\lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}}{\prod_{i=1}^n x_i!} = g(\underline{x} \mid \lambda) h(\underline{x})$, where $g(\underline{x} \mid \lambda) = \lambda^{\sum_{i=1}^n x_i} e^{-n\lambda}$ and $h(\underline{x}) = \dfrac{I_{\{0,1,2,\dots\}}(x_{(1)})}{\prod_{i=1}^n x_i!}$.

By the factorization theorem, $\sum_{i=1}^n X_i$ is sufficient for λ.

Now consider $0 < \lambda_1 < \lambda_2 < \infty$, where $\lambda_i \in \mathbb{R} : i = 1, 2$.

Then
$$\frac{g(t \mid \lambda_2)}{g(t \mid \lambda_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)^t \exp^{-n(\lambda_2 - \lambda_1)}.$$

Since this is an increasing function of t, then the poisson family has MLR. So, by the Karlin-Rubin Theorem, the test that rejects H_0 if and only if $\sum_{i=1}^n X_i = t \geq k(\lambda_0)$ is a UMP level α test, where $k(\lambda_0)$ satisfies $\alpha \geq P_{\lambda_0}(t \geq k(\lambda) + 1 \mid \lambda = \lambda_0)$.

So the acceptance region of this test is

$A(\lambda_0) = \{t : t > k(\lambda_0)\}$. Since $t \sim \text{Poisson}(n\lambda)$

we can choose $k(\lambda_0)$ such that $k(\lambda_0) = \text{Poisson}_{n\lambda, \alpha}$.

So then by the Gamma-Poisson relationship,

$\alpha \geq P(t \geq k(\lambda_0) + 1) = P(W \leq 2n\lambda)$, where $W \sim \text{Gamma}\left(\frac{2t+2}{2}, 2\right)$

So then $W \sim \chi^2_{2t+2}$. then $A(\lambda_0) = \{ t : \chi^2_{2t+2,\,1-\alpha} > 2n\lambda_0 \}$.

upper tail notation

Inverting this, we get the confidence region

$$\left\{ \lambda : 0 < \lambda < \frac{\chi^2_{2t+2,\,1-\alpha}}{2n} \right\}.$$

4) Given $X_1, \dots, X_n \overset{iid}{\sim} Exp(\lambda)$, and we have that

$$\pi(\lambda|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\left(\frac{1}{\lambda}\right)^{\alpha+1} e^{-1/\beta\lambda}, \quad 0<\lambda<\infty.$$

Then, in homework 6 we found the $1-\alpha$ credible set A, satisfying

$$P(\lambda \in A | y) = \int_A \pi(\lambda|y)\,d\lambda.$$

which is,

$$A: \{\lambda : 2(y-\beta)\chi^2_{\alpha-n+1,\frac{1-\alpha}{2}} < \lambda < 2(y-\beta)\chi^2_{\alpha-n+1,\frac{\alpha}{2}}\}.$$

We found the posterior density to be

$$\pi(\lambda|y) = \frac{(y-\beta)^{\alpha-n+1}}{\Gamma(\alpha-n+1)}\lambda^{\alpha-n+1} e^{-\lambda(y-\beta)}$$

which implies $\lambda|y \sim Gamma(\alpha-n+1, y-\beta)$. This is a unimodal density, so we can apply corollary 9.3.10 in Casella & Berger. So then the HPD region is the region that satisfies

$$1-\alpha = \int_{L(\lambda)}^{U(\lambda)} \pi(\lambda|y)\,d\lambda$$

which in homework 6, we found as A,

$$\{\lambda : 2(y-\beta)\chi^2_{\alpha-n+1,\frac{1-\alpha}{2}} < \lambda < 2(y-\beta)\chi^2_{\alpha-n+1,\frac{\alpha}{2}}\}.$$

5) We need to find a $1-\alpha$ Bayes HPD for σ^2 based on s^2, using the prior $\sigma^2 \sim IG(\alpha, \beta)$.

$$\Rightarrow \pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left\{-\frac{1}{\beta\sigma^2}\right\}, \quad 0 < \sigma^2 < \infty$$

We have that

$$\pi(s^2|\sigma^2) = \frac{f(s^2|\sigma^2)\pi(\sigma^2)}{m(s^2)}.$$

Since $s^2|\sigma^2 \sim \chi^2_{n-1}$ with sample size n, then we have that

$$\pi(s^2|\sigma^2) \propto \frac{1}{(\sigma^2)^{n-1/2}} \exp\left\{-\frac{(n-1)s^2}{2\sigma^2}\right\} \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left\{-\frac{1}{\beta\sigma^2}\right\}$$

$$= \frac{1}{(\sigma^2)^{n-1/2+\alpha+1}} \exp\left\{-\frac{(n-1)s^2}{2\sigma^2} + \frac{1}{\beta\sigma^2}\right\}$$

So $s^2|\sigma^2 \sim IG\left(\alpha + \frac{(n-1)}{2}, \left\{\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right\}^{-1}\right)$.

(Result also found in homework 2, 6(a).)

This distribution is unimodal, so then we can apply Corollary 9.3.10, where the Bayes HPD is given by

$$\Sigma : \left\{\sigma^2 : L(\sigma^2) \le \pi(\sigma^2|s^2) \le U(\sigma^2)\right\}$$

such that $L(\sigma^2) = a$, $U(\sigma^2) = b$ satisfies

$$\int_a^b \pi(\sigma^2|s^2)\, d\sigma^2 = 1-\alpha.$$

So then we want $\pi(a|s^2) = \pi(b|s^2)$, which implies

$$\Sigma : \left\{\sigma^2 : \frac{1}{a^{\left(\frac{n-1}{2}+\alpha\right)}} \exp\left\{-\frac{(n-1)s^2}{2a} + \frac{1}{\beta a}\right\} < \sigma^2 < \frac{1}{b^{\left(\frac{n-1}{2}+\alpha\right)}} \exp\left\{-\frac{(n-1)s^2}{2b} + \frac{1}{\beta b}\right\}\right\}$$

is a Bayes HPD credible set for σ^2.

6) $X \sim N(\mu, \sigma^2)$, σ^2 known $(n=1)$. Then we define the loss function

$$L(\theta, C) = b\,\text{Length}(C) - I_C(\theta), \ni I_C(\theta) = \begin{cases} 1 & \theta \in C \\ 0 & \theta \notin C \end{cases}.$$

a) To find the Risk function, you must find the expectation of $L(C, \theta)$. Let $C(x) = [x - c\sigma, x + c\sigma]$.

$$\begin{aligned} E[L(\mu, C)] &= bE[(x + c\sigma) - (x - c\sigma)] - E[I_C(\mu)] \\ &= b(2c\sigma) - P(x - c\sigma < \mu < x + c\sigma) \\ &= b2c\sigma - P\left(-c < \frac{x-\mu}{\sigma} < c\right) \\ &= b2c\sigma - P(-c < Z < c). \end{aligned}$$

b) To find $\frac{\partial}{\partial c} R(\mu, c)$, we have

$$\frac{\partial}{\partial c} R(\mu, c) = \frac{\partial}{\partial c}(b2c\sigma) - \frac{\partial}{\partial c} \int_{-c}^{c} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

by Fundamental Thm. of Calculus: $= 2b\sigma - \frac{2}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}$

* 2 in numerator comes from chain rule

c) Suppose $b\sigma > \frac{1}{\sqrt{2\pi}}$. Then

$$\frac{\partial}{\partial c} R(\mu, c) = 2b\sigma - \frac{2}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} > \frac{2}{\sqrt{2\pi}} - \frac{2}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} = \frac{2}{\sqrt{2\pi}}\left(1 - e^{-\frac{c^2}{2}}\right),$$

where $\left(\frac{2}{\sqrt{2\pi}}\right) > 0$ and $0 < e^{-c^2/2} \leq 1 \ \forall c$. So then $\forall c$, the derivative is positive and hence maximized where $e^{-\frac{c^2}{2}}$ is maximized, i.e., 0, as $\frac{\partial}{\partial c} R(\mu, c) = 0$ @ $c = 0$.

This implies that the "best interval" estimator is the point estimate if the variance is sufficiently small.

d) To minimize the risk we set $\frac{\partial}{\partial c} R(\mu, c) = 0$

Suppose $b_0 \le \frac{1}{\sqrt{2\pi}}$. Then $\frac{\partial}{\partial c} R(\mu, c)$ is not strictly positive across $c \ge 0$, and so the value of c that minimizes $R(\mu, c)$ is derived as.

$$0 \overset{set}{=} \frac{\partial}{\partial c} R(\mu, c) = 2b_0 - \frac{2}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}$$

$$\Rightarrow \quad b_0 = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}$$

$$\Rightarrow \quad \log\left(b_0\sqrt{2\pi}\right) = -\frac{c^2}{2}$$

$$\Rightarrow \quad c = \sqrt{-2\log\left(b_0\sqrt{2\pi}\right)}.$$

To show this is a <u>minimum</u>,

$$\frac{\partial^2}{\partial c^2} R(\mu, c) = \frac{2c}{\sqrt{2\pi}} e^{-\frac{c^2}{2}},$$

which is strictly positive across $c > 0$. So the c that minimizes $R(\mu, c)$ given $b_0 \le \frac{1}{\sqrt{2\pi}}$ is

$$c = \sqrt{-2\log\left(b_0\sqrt{2\pi}\right)}. \qquad \text{QED.}$$