Name: Carson Slater
Course: Theory of Statistics 2
Instructor: Dr. Harvill

Homework 3

1.

We are given $Y_1, Y_2, \ldots, Y_m \overset{iid}{\sim} \text{Bernoulli}(\theta)$. We want to test

$$H_0 \ : \ \theta \leq \theta_0 \quad \text{versus} \quad H_1 \ : \ \theta > \theta_0.$$

a.

Finding the MLE for θ, we find the global maximum for the log-likelihood function, $\ell(\theta)$. This has already been done in Homework 2 for this class but the work is shown again here.

$$\begin{aligned}
0 \overset{\text{set}}{=} \frac{\partial}{\partial\theta}\ell(\theta) &= \frac{\partial}{\partial\theta}\log\left(L(\theta|\boldsymbol{y})\right) \\
&= \frac{\partial}{\partial\theta}\log\left(\prod_{i=1}^{m}\theta^{y_i}(1-\theta)^{1-y_i}\right) \\
&= \frac{\partial}{\partial\theta}\log\left(\theta^{\sum_{i=1}^m y_i}(1-\theta)^{m-\sum_{i=1}^m y_i}\right) \\
&= \frac{\partial}{\partial\theta}\left(\left(\sum_{i=1}^m y_i\right)\log(\theta) + \left(m - \sum_{i=1}^m y_i\right)\log(1-\theta)\right) \\
&= \frac{\sum_{i=1}^m y_i}{\theta} - \frac{m - \sum_{i=1}^m y_i}{1-\theta} \\
\iff \frac{1-\theta}{\theta} &= \frac{m - \sum_{i=1}^m y_i}{\sum_{i=1}^m y_i} \\
\iff \frac{1-\theta}{\theta} &= \frac{1 - \frac{1}{m}\sum_{i=1}^m y_i}{\frac{1}{m}\sum_{i=1}^m y_i} \\
\iff \frac{1-\theta}{\theta} &= \frac{1-\bar{y}}{\bar{y}} \\
\implies \theta &= \bar{y}.
\end{aligned}$$

To show $\theta = \bar{Y}$ is a global maximum, we can show that $\frac{\partial^2}{\partial\theta^2}\ell(\theta)\Big|_{\theta=\hat{\theta}} < 0$ for all $\hat{\theta} \in [0,1]$.

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2}\ell(\theta)\Big|_{\theta=\hat{\theta}} &= \frac{\partial}{\partial\theta}\left(\frac{\sum_{i=1}^m y_i}{\theta} - \frac{m - \sum_{i=1}^m y_i}{1-\theta}\right)\Big|_{\theta=\hat{\theta}} \\
&= -\frac{\sum_{i=1}^m y_i}{\theta^2} + \frac{m - \sum_{i=1}^m y_i}{(1-\theta)^2}\Big|_{\theta=\hat{\theta}} < 0 \ \forall \ |\hat{\theta}|[0,1].
\end{aligned}$$

Because $\left|\frac{\sum_{i=1}^m y_i}{\theta^2}\right| > \left|\frac{m - \sum_{i=1}^m y_i}{(1-\theta)^2}\right| \ \forall \ \theta \in [0,1]$, we know that $\frac{\partial^2}{\partial\theta^2}\ell(\theta)\Big|_{\theta=\hat{\theta}} < 0$ over the viable values of θ. So $\hat{\theta} = \bar{Y}$.

b.

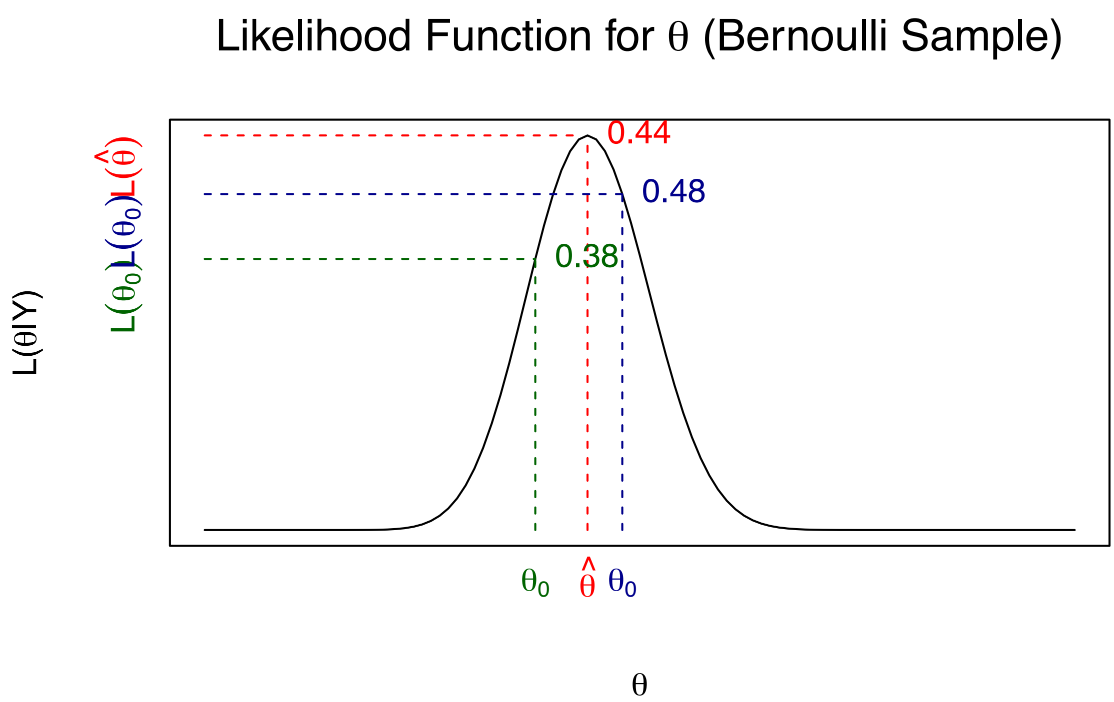
Under H_0, we suppose that $\Theta_0 = \{0 \leq \theta \leq \hat{\theta} < 1\}$. Finding the restricted MLE under the null hypothesis would require restricting θ to $[0, \theta_0]$. Since the domain of the parameter is restricted to a subset of the support for $Y_1, \ldots, Y_m$, we know that $\ell(\theta)$ is the same, just it is now over this new restricted support. We know from part (a) that $\ell(\theta)$ is maximized at $\hat{\theta} = \bar{Y}$. Since $\ell(\theta)$ is concave down on $[0, 1]$, we know that is it also concave down on $[0, \theta]$. So the restricted MLE of θ is still $\bar{Y}$ unless $\bar{Y} > \theta_0$. So the restricted MLE of θ on $[0, \theta_0]$ is $\hat{\theta}_0 = \min(\bar{Y}, \theta_0)$.

c.

```
plot(theta_values, likelihood_values, type = "l", col = "black",
     xaxt = "n", yaxt = "n", xlab = latex2exp::TeX(r'(\theta)'),
     ylab = latex2exp::TeX(r'(L(\theta|Y))'),
     main = latex2exp::TeX(r'(Likelihood Function for \theta (Bernoulli Sample))'),
     cex.main = 1.3)

# Mark the maximum likelihood estimate (MLE)
mle <- theta_values[which.max(likelihood_values)]
lines(x = rep(mle, times = 44),
      y = seq(0, max(likelihood_values), length.out = 44),
      col = "red", lty = 2)
lines(x = theta_values[which(theta_values < 0.44)],
      y = rep(max(likelihood_values),
              times = length(theta_values[which(theta_values < 0.44)])),
      col = "red", lty = 2)
llh_theta_0 <- ber_likelihood(theta_0, data)
lines(x = theta_values[which(theta_values <= theta_0)],
      y = rep(llh_theta_0, times = length(theta_values[which(theta_values <= theta_0)])),
      col = "darkgreen", lty = 2)
lines(x = rep(theta_0, times = 44),
      y = seq(0, llh_theta_0, length.out = 44),
      col = "darkgreen", lty = 2)
llh_theta_greater <- ber_likelihood(0.48, data)
lines(x = theta_values[which(theta_values <= 0.48)],
      y = rep(llh_theta_greater, times = length(theta_values[which(theta_values <= 0.48)])),
      col = "darkblue", lty = 2)
lines(x = rep(0.48, times = 44),
      y = seq(0, llh_theta_greater, length.out = 44),
      col = "darkblue", lty = 2)
text(mle, max(likelihood_values), round(mle, 2), pos = 4, col = "red")
text(theta_0, llh_theta_0, theta_0, pos = 4, col = "darkgreen")
text(0.48, llh_theta_greater, 0.48, pos = 4, col = "darkblue")
mtext(expression(hat(theta)), side = 1, line = 0.5, col = "red", at = c(mle))
mtext(expression(L(hat(theta))), side = 2, line = 0.5, col = "red", at = max(likelihood_valu
mtext(expression(theta[0]), side = 1, line = 0.5, col = "darkgreen", at = c(theta_0))
```

```
mtext(expression(L(theta[0])), side = 2, line = 0.5, col = "darkgreen", at = llh_theta_0, ad
mtext(expression(theta[0]), side = 1, line = 0.5, col = "darkblue", at = 0.48)
mtext(expression(L(theta[0])), side = 2, line = 0.5, col = "darkblue", at = llh_theta_greate
```



d.

Finding the likelihood test statistic, we take the ratio of the likelihood function under the null hypothesis over the unrestricted likelihood function.

$$\begin{aligned}
\lambda(\boldsymbol{y}) &= \frac{L(\theta_0|\boldsymbol{y})}{L(\hat{\theta}|\boldsymbol{y})} \\
&= \frac{(\theta_0)^{\sum_{i=1}^m y_i}(1-\theta_0)^{m-\sum_{i=1}^m y_i}}{(\hat{\theta})^{\sum_{i=1}^m y_i}(1-\hat{\theta})^{m-\sum_{i=1}^m y_i}} \\
&= \left[\frac{\theta_0(1-\hat{\theta})}{\hat{\theta}(1-\theta_0)}\right]^{\sum_{i=1}^m y_i}\left(\frac{1-\theta_0}{1-\hat{\theta}}\right)^m.
\end{aligned}$$

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Substituting in $\theta_0 = \hat{\theta}_0 = \min(\bar{Y}, \theta_0)$ and $\hat{\theta} = \bar{Y}$,

$$\begin{aligned}\lambda(\boldsymbol{y}) &= \left[\frac{\min(\bar{y},\theta_0)(1-\bar{y})}{\bar{y}(1-\min(\bar{y},\theta_0))}\right]^{\sum_{i=1}^m y_i}\left(\frac{1-\min(\bar{y},\theta_0)}{1-\bar{y}}\right)^m \\ &= \begin{cases}\left[\dfrac{\theta_0(1-\bar{y})}{\bar{y}(1-\theta_0)}\right]^{\sum_{i=1}^m y_i}\left(\dfrac{1-\theta_0}{1-\bar{y}}\right)^m & \theta_0 < \hat{\theta}, \\ 1 & \theta_0 \geq \hat{\theta}.\end{cases}\end{aligned}$$

The likelihood ratio test is the test that rejects H_0 for $\{\boldsymbol{y} \, : \, \lambda(\boldsymbol{y}) \leq c\}$ where $0 \leq c \leq 1$.

e.

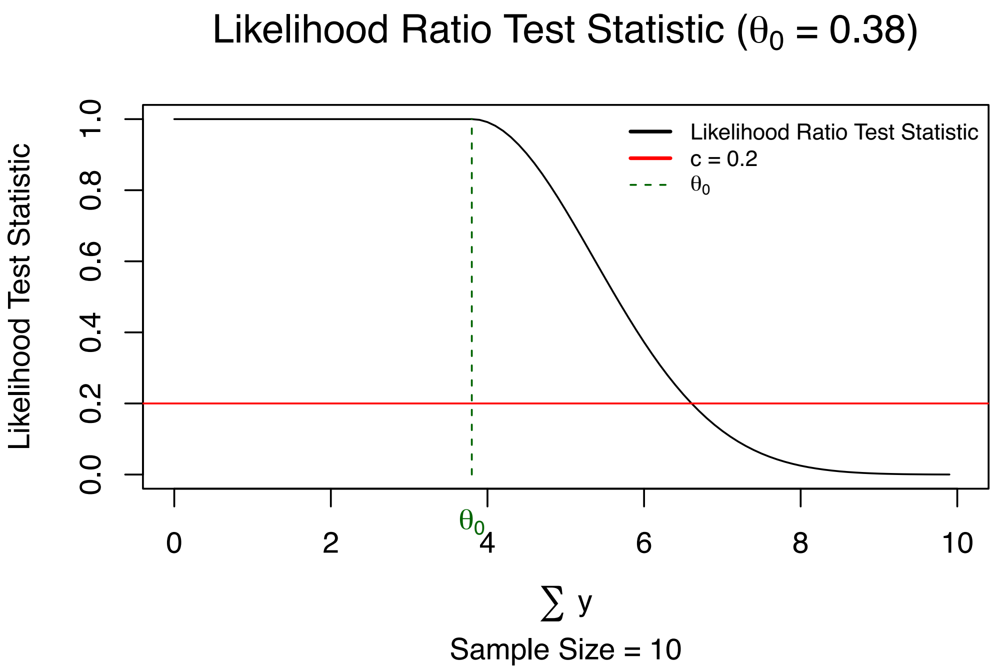
```
sum_y <- seq(3.8, 10, 0.1)

sample_size <- c(10)

lambda <- function(n) {
  ((theta_0*(1-(sum_y/n)))/((sum_y/n)*(1- theta_0)))^(sum_y)*((1 - theta_0)/(1-(sum_y/n)))^r
}

ber_lambda <-lambda(sample_size)

plot(sum_y, ber_lambda, type = "l", col = "black",
     xlab = latex2exp::TeX(r'(\sum y)'),
     ylab = "Likelihood Test Statistic",
     xlim = c(0, 10),
     main = latex2exp::TeX(r'(Likelihood Ratio Test Statistic ($\theta_0$ = 0.38))'),
     cex.main = 1.3)
title(sub = "Sample Size = 10")
lines(seq(0, 3.8, 0.1), rep(1, length(seq(0, 3.8, 0.1))), col = "black")
lines(x = rep(sample_size*theta_0, times = 44),
      y = seq(0, 1, length.out = 44),
      col = "darkgreen", lty = 2)
abline(h = 0.2, col = "red")
mtext(expression(theta[0]), side = 1, line = 0.5, col = "darkgreen", at = c(sample_size*thet
legend("topright", c("Likelihood Ratio Test Statistic",
                     "c = 0.2",
                     expression(theta[0])),
       col = c("black", "red", "darkgreen"),
       lwd = c(2, 2, 1),
       lty = c(1, 1, 2),
       bty = "n",
       cex = 0.75)
```



f.

If $T(\boldsymbol{Y})$ is a sufficient statistic for θ and $\lambda^*(t)$ and $\lambda(\boldsymbol{y})$ are LRT statistics based on $T(\boldsymbol{Y})$ and $\boldsymbol{Y}$ respectively, $\lambda^*(T(\boldsymbol{Y})) = \lambda(\boldsymbol{Y})$ for every $\boldsymbol{y} \in \mathcal{Y}$. Hence we can show that if $T(\boldsymbol{Y}) = \bar{Y}$, then

$$\lambda^*(\bar{y}) = \left[\frac{\theta_0(1-\bar{y})}{\bar{y}(1-\theta_0)}\right]^{m\bar{y}} \left(\frac{1-\theta_0}{1-\bar{y}}\right)^m.$$

We can show this function is monotone decreasing by taking the derivative,

$$\begin{aligned}
\log(\lambda^*(\bar{y})) &= m\bar{y}\log(\theta_0) + m\log(1-\theta_0) - m\bar{y}\log(1-\theta_0) \\
&\quad - m\bar{y}\log(\bar{y}) - m\log(1-\bar{y}) - m\bar{y}\log(1-\bar{y}) \\
\implies \frac{\partial}{\partial\bar{y}}\log(\lambda^*(\bar{y})) &= m\log(\theta_0) - m\log(1-\theta_0) - m\log(\bar{y}) - \frac{m\bar{y}}{\bar{y}} - \left(\frac{m}{1-\bar{y}} + m\log(1-\bar{y}) + \frac{m\bar{y}}{1-\bar{y}}\right) \\
&= m\log\left(\frac{\theta_0}{(1-\theta_0)\bar{y}}\right) + m - \frac{m(1-\bar{y})}{(1-\bar{y})} - m\log(1-\bar{y}) \\
\implies \exp\left\{\frac{\partial}{\partial\bar{y}}\log(\lambda^*(\bar{y}))\right\} &= \left(\frac{\theta_0}{\bar{y}(1-\theta_0)(1-\bar{y})}\right)^m = \frac{\partial}{\partial\bar{y}}\lambda^*(\bar{y}).
\end{aligned}$$

Full disclosure, I did not take this derivative correctly. It was nasty. Here was my good faith effort. I know it is monotonically decreasing per my plot, but could not show analytically.
Which is a function that is monotonically decreasing over $T(\boldsymbol{Y}) = \bar{Y}$. So since $\bar{Y}$ is a sufficient statistic for θ, then $\lambda^*(T(\boldsymbol{Y})) = \lambda(\boldsymbol{Y})$ for all $\boldsymbol{y} \in \mathcal{Y}$. So the LRT is the test that rejects H_0 such that $\{\bar{y} \,:\, \lambda^*(t) \le c\}$ where $0 \le c \le 1$, which implies that the LRT is the also test that rejects H_0 such that $\{\bar{y} \,:\, t \ge \lambda^{-1\,*}(c) = k\}$ (inequality flips because inverse fn.) where $0 \le k \le 1$ and $t = \bar{y}$.

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2.

We are given $X_1, \ldots, X_n \overset{iid}{\sim} \text{Pareto}(\nu, \theta)$, with PDF

$$f(x|\nu,\theta) = \frac{\theta\nu^\theta}{x^{\theta+1}}, \quad \nu < x < \infty, \quad \nu > 0,\ \theta > 0.$$

a.

We use the Factorization Theorem to find a sufficient statistic for (ν, θ).

$$\begin{aligned} f(\boldsymbol{x}|\nu,\theta) &= \prod_{i=1}^n \frac{\theta\nu^\theta}{x_i^{\theta+1}} I_{[\nu,\infty]}(x_i) \\ &= \frac{(\theta\nu^\theta)^n}{\prod_{i=1}^n x_i^{\theta+1}} I_{[\nu,\infty]}(\min(x_i)) \\ &= g(T(\boldsymbol{x})|\nu,\theta)h(\boldsymbol{x}) \end{aligned}$$

where $g(T(\boldsymbol{x})|\nu,\theta) = f(\boldsymbol{x}|\nu,\theta)$, and $h(\boldsymbol{x}) = 1$. So then by the Factorization Theorem, $T(\boldsymbol{x}) = (X_{(1)}, \prod_{i=1}^n X_i)$ is a sufficient statistic for (ν,θ), where $X_{(1)} = \min(X_i)$.

b.

The likelihood function for ν is

$$L(\nu|\theta=\theta,\boldsymbol{x}) = \frac{(\theta\nu^\theta)^n}{\prod_{i=1}^n x_i^{\theta+1}} I_{[\nu,\infty]}(\min(x_i)),$$

which is a monotone increasing function over ν. Since $0 < \nu < x_{(1)}$, the value that maximizes $L(\nu|\theta=\theta,\boldsymbol{x})$ with respect to ν is $X_{(1)}$. Hence the MLE for ν, $\hat{\nu} = X_{(1)}$. As for the MLE of θ, the log-likelihood function, $\ell(\nu,\theta)$, for $\boldsymbol{x}$ is

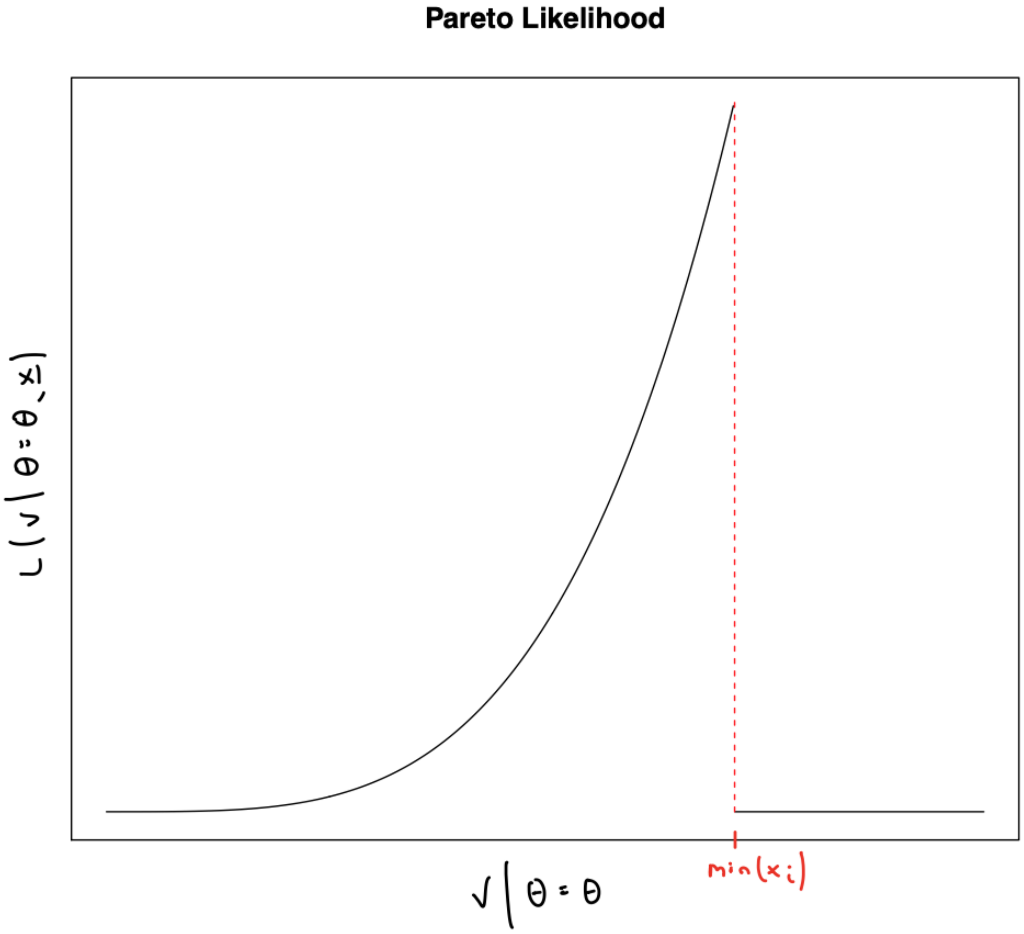
$$\ell(\nu,\theta) = n\log(\theta) + n\theta\log(\nu) - (\theta+1)\sum_{i=1}^n \log(x_i)$$

Finding the global maximum, we can set $\frac{\partial}{\partial\theta}\ell(\nu,\theta) = 0$.

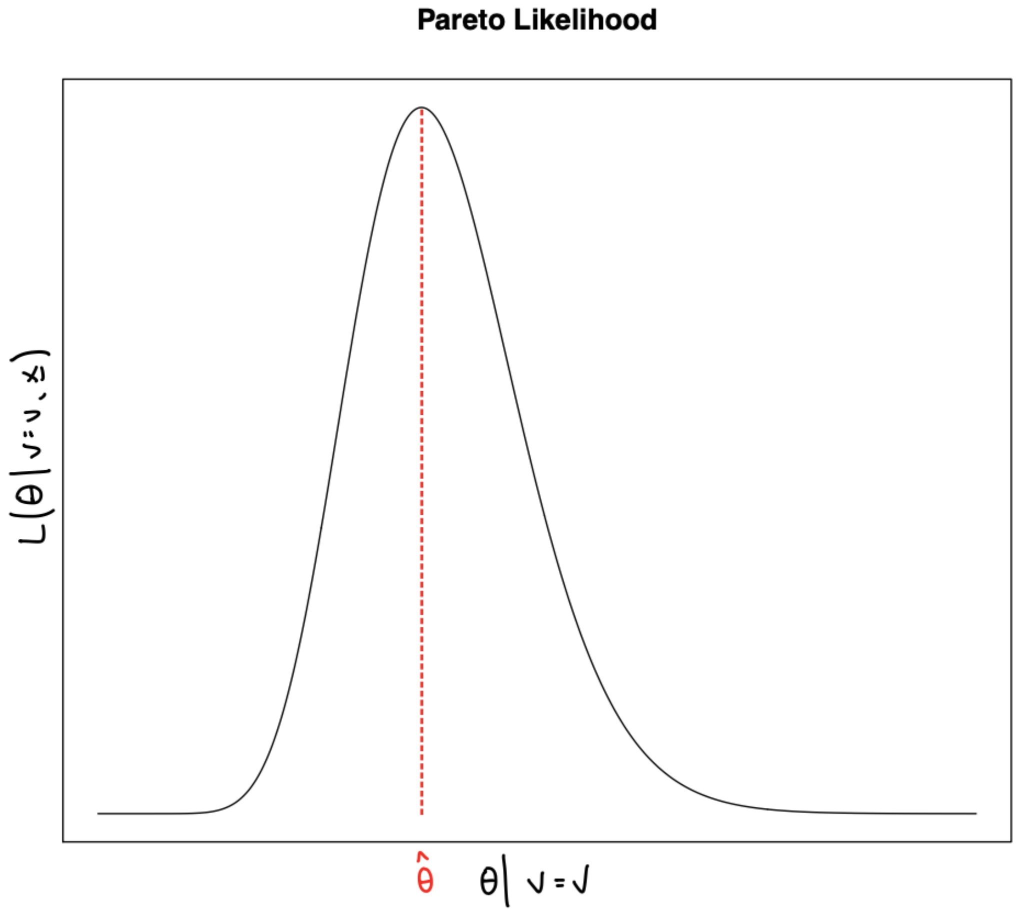
$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta}\ell(\nu,\theta) \\ &= \frac{n}{\theta} + n\log(\nu) - \sum_{i=1}^n \log(x_i) \\ \iff \sum_{i=1}^n \log(x_i) &= \frac{n}{\theta} + n\log(\nu) \\ \iff \hat{\theta} &= \frac{n}{\log(\sum_{i=1}^n (x_i)/\hat{\nu}^n)} \\ \iff \hat{\theta} &= \frac{n}{\log(\sum_{i=1}^n (X_i)/X_{(1)}^n)}. \end{aligned}$$

c.

Since $0 < \nu < x_{(1)}$, the value that maximizes $L(\nu|\theta = \theta, \boldsymbol{x})$ with respect to ν is $X_{(1)}$. Hence the MLE for ν, $\hat{\nu} = X_{(1)}$.



d.



e.

With ν unknown, we want to test

$$H_0 \ : \ \theta = 1 \quad \text{versus} \quad H_1 \ : \ \theta \neq 1.$$

We know that

$$\hat{\theta} = \frac{n}{\log(\sum_{i=1}^n (X_i)/X_{(1)}^n)}.$$

We write the likelihood ratio test statistic in terms of $T(\boldsymbol{X}) = \log\left(\frac{\sum_{i=1}^n (X_i)}{X_{(1)}^n}\right) = t$. Finding the likelihood ratio test statistic,

$$\begin{aligned}
\lambda(\boldsymbol{x}) &= \frac{L(1|\boldsymbol{x})}{L(\hat{\theta}|\boldsymbol{x})} \\
&= \frac{\hat{\nu}^n \left(\prod_{i=1}^n x_i\right)^{\frac{n}{t}+1}}{\left(\frac{n}{t}\hat{\nu}\right)^{\frac{n}{t}} \left(\prod_{i=1}^n x_i\right)^2} \\
&= \left(\frac{t}{n}\right)^n \left(x_{(1)}^{\left(1-\frac{n}{t}\right)}\right)^n \left(\prod_{i=1}^n x_i\right)^{\frac{n}{t}-1} \\
&= \left(\left(\frac{t}{n}\right)^n \left(\frac{x_{(1)}}{\prod_{i=1}^n x_i^{\frac{1}{n}}}\right)^{\left(1-\frac{n}{t}\right)}\right)^n \\
&= \left(\frac{t}{n}\right)^n \left(\frac{x_{(1)}^n}{\prod_{i=1}^n x_i}\right)^{\left(1-\frac{n}{t}\right)} \\
&= \left(\frac{t}{n}\right)^n e^{n-t} \qquad \left(\text{Since } \left(\frac{x_{(1)}^n}{\prod_{i=1}^n x_i}\right) = e^{-t}\right)
\end{aligned}$$

which is a function that is not monotone increasing. Since n is fixed, when $n > t$, the function is decreasing over t, and when $n < t$, the function is increasing over t. Hence the LRT has a critical region of the form $\{\boldsymbol{x} \ : \ T(\boldsymbol{x}) < c_1 \text{ or } T(\boldsymbol{x}) > c_2\}$ such that $0 \leq c_1 < c_2 \leq 1$.

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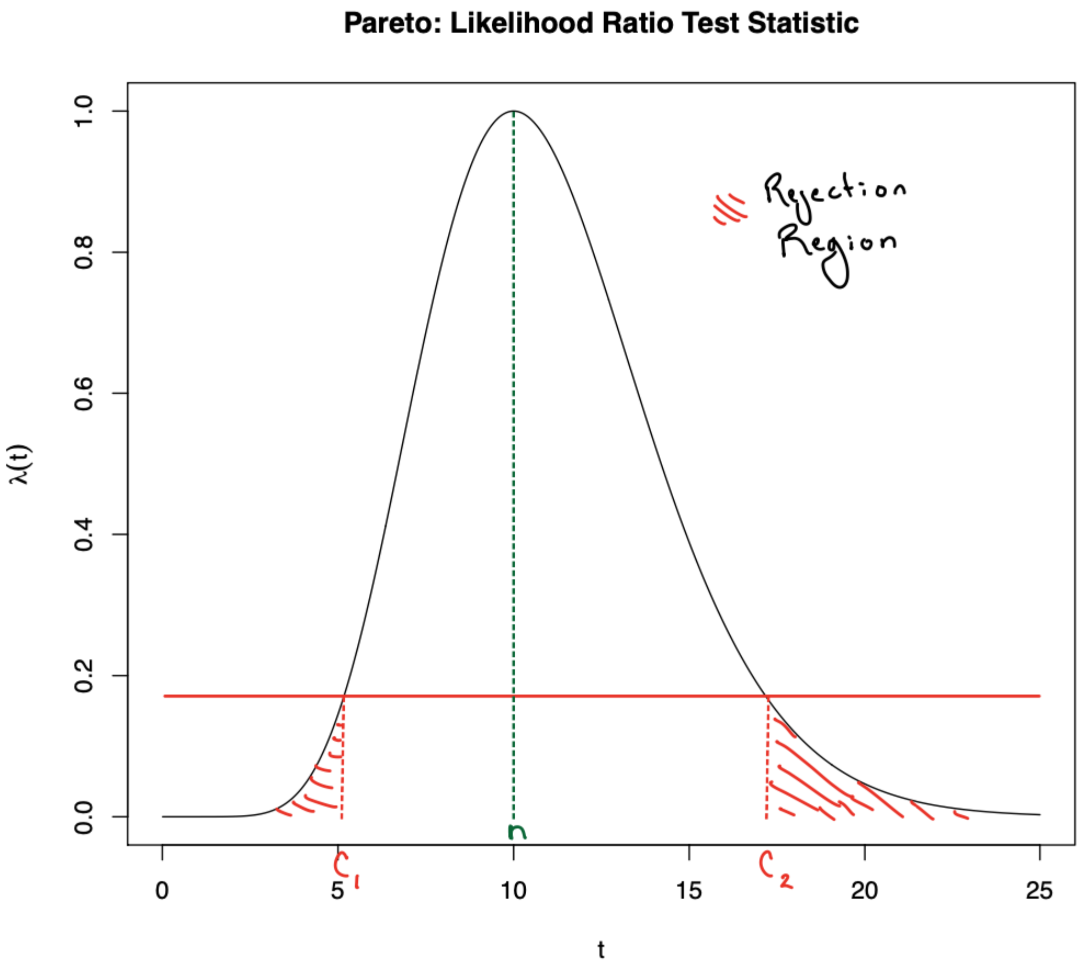
f.



3.

We are given $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$, where σ^2 is known. We consider testing

$$H_0 \ : \ \theta \leq \theta_0 \quad \text{versus} \quad H_1 \ : \ \theta > \theta_0.$$

We know that the likelihood function for an IID normal sample is

$$L(\theta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \theta)^2}{2\sigma^2}\right),$$

yielding the likelihood ratio test statistic

$$\begin{aligned}
\lambda(\boldsymbol{x}) &= \frac{L(\theta_0, \sigma^2)}{L(\hat{\theta}, \sigma^2)} \\
&= \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 - \sum_{i=1}^{n}(x_i - \theta_0)^2\right)\right\} \\
&= \exp\left\{-\frac{1}{2\sigma^2}\left(-\sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \theta_0)^2 + \sum_{i=1}^{n}(x_i - \bar{x})\right)\right\} \\
&= \exp\left\{\left(-\frac{1}{2\sigma^2}\right)\left(+n(\bar{x} - \theta_0)^2\right)\right\} \\
&= \exp\left\{\frac{-n(\bar{x} - \theta_0)^2}{2\sigma^2}\right\} \\
\implies \lambda(\boldsymbol{x}) &= \begin{cases} \exp\left\{\frac{-n(\bar{x} - \theta_0)^2}{2\sigma^2}\right\}, & \bar{x} < \theta_0 \\ 1, & \bar{x} \geq \theta_0. \end{cases}
\end{aligned}$$

The likelihood ratio test is the test that rejects H_0 for $\{\boldsymbol{x} \ : \ \lambda(\boldsymbol{x}) \leq c\}$ where $0 \leq c \leq 1$.

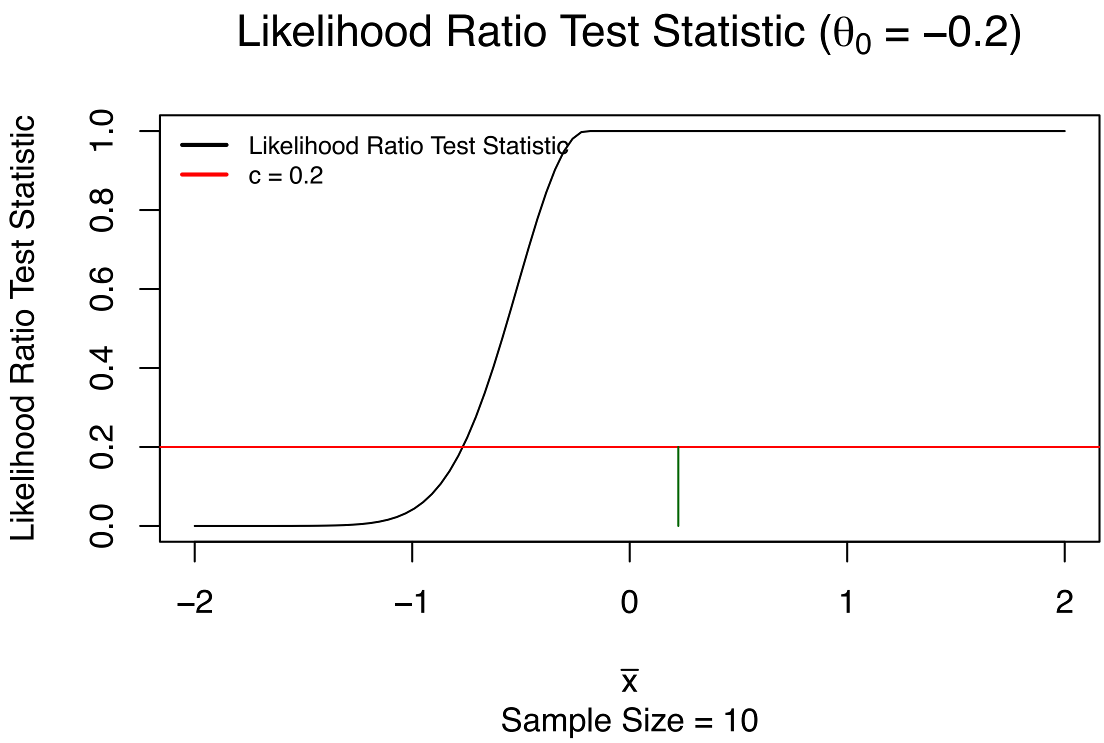
```r
xbar <- seq(-2, 2, length.out = 100)

# function computes LRT
lrt <- function(xbar, theta_0, sigma_sq, n) {
  output <- exp(-n*(xbar - theta_0)^2/(2*sigma_sq))
  for (i in seq_along(xbar)) {
    if (xbar[i] >= theta_0) output[i] <- 1
  }
  output
}

lrt <- lrt(xbar, theta_0 = -0.2, sigma_sq = 1, n = 10)

# plot
plot(xbar, lrt, type = "l", col = "black",
     xlab = latex2exp::TeX(r'(\bar{x})'),
```

```
     ylab = "Likelihood Ratio Test Statistic",
     xlim = c(-2, 2),
     main = latex2exp::TeX(r'(Likelihood Ratio Test Statistic ($\theta_0$ = -0.2))'),
     cex.main = 1.3)
title(sub = "Sample Size = 10")
abline(h = 0.2, col = "red")
lines(x = rep(lrt[32], length = 10), y = seq(0, 0.2, length.out = 10), col = "darkgreen")
legend("topleft", c("Likelihood Ratio Test Statistic",
                    "c = 0.2"),
       col = c("black", "red"),
       lwd = c(2, 2),
       lty = c(1, 1),
       bty = "n",
       cex = 0.75)
```



The rejection region is any region where $\lambda(\boldsymbol{x}) \leq c$, where $c = 0.2$ is denoted by the red line in the figure.

4.

a.

We are given $Y \sim \text{IG}(\alpha, \beta)$. Let $W = g(Y) = \dfrac{1}{Y}$. We then show that $W \sim \text{Gamma}(\alpha, \beta)$.

$$\begin{aligned} f_W w &= f_Y \frac{1}{w} \left| \frac{d}{dw} \frac{1}{w} \right|, \quad 0 < \frac{1}{w} < \infty \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(w^{-1})^{\alpha+1}} \exp\left\{ -\frac{1}{\beta(w^{-1})} \right\}, \quad 0 < w < \infty \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} w^{\alpha+1} \exp\left\{ -\frac{w}{\beta} \right\}, \quad 0 < w < \infty \\ \implies & W \sim \text{Gamma}(\alpha, \beta). \end{aligned}$$

b.

This is the same problem as Homework 2 problem 6(a).

We have that $S^2|\sigma^2 \sim \chi^2_{n-1}$ with sample size n. This is equivalent to saying that $S^2|\sigma^2 \sim \text{Gamma}\left(\dfrac{n-1}{2}, \dfrac{2\sigma^2}{n-1}\right)$. We suppose the prior distribution $\pi(\sigma^2)$ is an inverse gamma with shape parameter α and scale parameter β,

$$\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left\{ -\frac{1}{\beta\sigma^2} \right\}, \quad 0 < \sigma^2 < \infty.$$

We know that the posterior distribution for $\sigma^2|S^2$ is

$$\pi(\sigma^2|S^2) = \frac{f(S^2|\sigma^2)\pi(\sigma^2)}{m(S^2)}.$$

Knowing $f(S^2|\sigma^2)$ and $\pi(\sigma^2)$, we can say that

$$\begin{aligned} \pi(\sigma^2|S^2) &\propto \frac{1}{(\sigma^2)^{(n-1)/2}} \exp\left\{ -\frac{(n-1)S^2}{2\sigma^2} \right\} \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left\{ -\frac{1}{\beta\sigma^2} \right\} \\ &= \frac{1}{(\sigma^2)^{(n-1)/2+\alpha+1}} \exp\left\{ -\left(\frac{(n-1)S^2}{2\sigma^2} + \frac{1}{\beta\sigma^2} \right) \right\} \\ &= \text{IG}\left(\alpha + \frac{(n-1)}{2}, \left\{ \frac{(n-1)S^2}{2} + \frac{1}{\beta} \right\}^{-1} \right). \end{aligned}$$

This implies that the kernel of the posterior is proportional to an $\text{IG}\left(\alpha + \frac{(n-1)}{2}, \left\{ \frac{(n-1)S^2}{2} + \frac{1}{\beta} \right\}^{-1} \right)$. So $\sigma^2|S^2 \sim \text{IG}\left(\alpha + \frac{(n-1)}{2}, \left\{ \frac{(n-1)S^2}{2} + \frac{1}{\beta} \right\}^{-1} \right)$.

c.

Here we consider

$$H_0 : \sigma \leq 1 \quad \text{versus} \quad H_1 : \sigma > 1.$$

To find the region of the sample space for which a Bayes test will decide $\sigma \leq 1$, we consider a transformation of the posterior $\pi(\sigma^2|S^2)$,

$$W = 2(\sigma^2)^{-1}\left(\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right).$$

Note the Jacobean for this transformation is $|\mathbf{J}| = 2\left(\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right)^{-2}$. (See following pages)

c) $P(\sigma \le 1 | S^2 = s^2) > P(\sigma > 1 | S^2 = s^2)$

$$\pi(\sigma^2 | s^2) = \frac{1}{(\sigma^2)^{\left(\frac{n-1}{2}\right)+\alpha+1}} \exp\left\{-\frac{1}{w}\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)\right\}$$

$$\left|\frac{dw}{d\sigma^2}\right| = \left|-2\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)(\sigma^2)^{-2}\right| =$$

Let $W = 2(\sigma^2)^{-1}\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)$

$e^{-\frac{x}{\beta}}$ scale

$e^{-\lambda x}$ rate

$$f_w(w) = f_y(g^{-1}(\sigma^2)) \left|\frac{\partial g(\sigma^2)}{\partial \sigma^2}\right|$$

$$\frac{\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)^{\alpha+\frac{(n-1)}{2}}}{\Gamma\left(\alpha+\frac{(n-1)}{2}\right)} \frac{1}{(w)^{\left(\frac{n-1}{2}\right)+\alpha+1}} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)\right\} |J|$$

$W \sim \chi^2$

$$= \frac{\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)^{\alpha+\frac{(n-1)}{2}} (\sigma^2)^{\left(\frac{n-1}{2}\right)+\alpha+1}}{\Gamma\left(\alpha+\frac{(n-1)}{2}\right)\left(2\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)\right)^{\left(\frac{n-1}{2}\right)+\alpha+1}} \exp\left\{-\frac{\sigma^2}{2} \cdot \frac{\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)}{\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)}\right\} |J|$$

adding the Jacobean, $|J| = 2\left(\frac{(n-1)s^2}{2} + \frac{1}{\beta}\right)$, we have

$$= \frac{1}{\Gamma\left(\alpha+\frac{(n-1)}{2}\right) 2^{\left(\frac{n-1}{2}\right)+\alpha}} (\sigma^2)^{\left(\frac{n-1}{2}\right)+\alpha-1} \exp\left\{-\frac{\sigma^2}{2}\right\}$$

which follows a $\chi^2_{n-1+2\alpha}$

Rejection region is where

$$P(\sigma > 1 \mid S^2 = s^2) = P(\sigma^2 > 1 \mid S^2 = s^2) > 0.5.$$

$$= P\left(2(\sigma^2)^{-1}\left(\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right) > (n-1)s^2 + \frac{1}{2\beta} \,\middle|\, S^2 = s^2\right)$$

$$= P\left(W > (n-1)s^2 + \frac{1}{2\beta} \,\middle|\, S^2 = s^2\right) > 0.5$$

$$= 1 - \underbrace{P\left(W \le (n-1)s^2 + \frac{1}{2\beta} \,\middle|\, S^2 = s^2\right)}_{\chi^2_{n-1+2\alpha}} \ge 0.5$$

d) $H_0: \sigma \le 1$ $H_a: \sigma > 1$

LRT, given $\mu = \bar{x}$, and $\hat{\sigma}^2 = (n-1)s^2/n$

$$\lambda(\underline{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta \mid \underline{x})}{\sup_{\theta \in \Theta} L(\theta \mid \underline{x})}$$

$$\lambda(\underline{x}) = \frac{L(\sigma_0^2 \mid \underline{x}, \bar{x})}{L(\hat{\sigma}^2 \mid \underline{x}, \bar{x})} = \frac{\left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} \exp\left\{-(2\sigma_0^2)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\right\}}{\left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{n/2} \exp\left\{-(2\hat{\sigma}^2)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2\right\}}$$

$$= \frac{\left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} \exp\left\{-\frac{1}{(2\sigma_0^2)} \sum_{i=1}^{n} (x_i - \bar{x})^2\right\}}{\left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{n/2} \exp\left\{-\frac{1}{(2\hat{\sigma}^2)} \sum_{i=1}^{n} (x_i - \bar{x})^2\right\}}$$

$$\frac{(n-1)}{n} \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left\{\sum_{i=1}^{n} (x_i - \bar{x})^2 \left(\frac{1}{2\hat{\sigma}^2} - \frac{1}{2\sigma_0^2}\right)\right\}$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left\{-\frac{n}{2} \frac{\hat{\sigma}^2}{\sigma_0^2}\right\}$$

$$\sigma_0^2 = \begin{cases} \sigma_0^2 & \text{if } \sigma_0 < \frac{n-1}{n} S^2 \\ \frac{n-1}{n} S^2 & \text{if } \sigma_0 \geq \frac{n-1}{n} S^2 \end{cases}$$

$$\lambda(\underline{x}) = \begin{cases} \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left\{-\frac{n}{2} \frac{\hat{\sigma}^2}{\sigma_0^2}\right\}, & \sigma_0 < \frac{n-1}{n} S^2 \\ 1, & \sigma_0 \geq \frac{n-1}{n} S^2 \end{cases}$$