



Homework 3

1.

We are given $Y_1, Y_2, \dots, Y_m \stackrel{iid}{\sim} \text{Bernoulli}(\theta)$. We want to test

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

a.

Finding the MLE for θ , we find the global maximum for the log-likelihood function, $\ell(\theta)$. This has already been done in Homework 2 for this class but the work is shown again here.

$$\begin{aligned} 0 &\stackrel{\text{set}}{=} \frac{\partial}{\partial \theta} \ell(\theta) = \frac{\partial}{\partial \theta} \log(L(\theta|\mathbf{y})) \\ &= \frac{\partial}{\partial \theta} \log\left(\prod_{i=1}^m \theta^{y_i} (1-\theta)^{1-y_i}\right) \\ &= \frac{\partial}{\partial \theta} \log\left(\theta^{\sum_{i=1}^m y_i} (1-\theta)^{m-\sum_{i=1}^m y_i}\right) \\ &= \frac{\partial}{\partial \theta} \left(\left(\sum_{i=1}^m y_i\right) \log(\theta) + \left(m - \sum_{i=1}^m y_i\right) \log(1-\theta) \right) \\ &= \frac{\sum_{i=1}^m y_i}{\theta} - \frac{m - \sum_{i=1}^m y_i}{1-\theta} \\ \Leftrightarrow \frac{1-\theta}{\theta} &= \frac{m - \sum_{i=1}^m y_i}{\sum_{i=1}^m y_i} \\ \Leftrightarrow \frac{1-\theta}{\theta} &= \frac{1 - \frac{1}{m} \sum_{i=1}^m y_i}{\frac{1}{m} \sum_{i=1}^m y_i} \\ \Leftrightarrow \frac{1-\theta}{\theta} &= \frac{1-\bar{y}}{\bar{y}} \\ \Rightarrow \theta &= \bar{y}. \end{aligned}$$

To show $\theta = \bar{Y}$ is a global maximum, we can show that $\frac{\partial^2}{\partial \theta^2} \ell(\theta) \Big|_{\theta=\hat{\theta}} < 0$ for all $\hat{\theta} \in [0, 1]$.

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2} \ell(\theta) \Big|_{\theta=\hat{\theta}} &= \frac{\partial}{\partial \theta} \left(\frac{\sum_{i=1}^m y_i}{\theta} - \frac{m - \sum_{i=1}^m y_i}{1-\theta} \right) \Big|_{\theta=\hat{\theta}} \\ &= -\frac{\sum_{i=1}^m y_i}{\theta^2} + \frac{m - \sum_{i=1}^m y_i}{(1-\theta)^2} \Big|_{\theta=\hat{\theta}} < 0 \quad \forall \hat{\theta} \in [0, 1]. \end{aligned}$$

Because $\left| \frac{\sum_{i=1}^m y_i}{\theta^2} \right| > \left| \frac{m - \sum_{i=1}^m y_i}{(1-\theta)^2} \right| \quad \forall \theta \in [0, 1]$, we know that $\frac{\partial^2}{\partial \theta^2} \ell(\theta) \Big|_{\theta=\hat{\theta}} < 0$ over the viable values of θ . So $\hat{\theta} = \bar{Y}$.



b.

Under H_0 , we suppose that $\Theta_0 = \{0 \leq \theta \leq \hat{\theta} < 1\}$. Finding the restricted MLE under the null hypothesis would require restricting θ to $[0, \theta_0]$. Since the domain of the parameter is restricted to a subset of the support for Y_1, \dots, Y_m , we know that $\ell(\theta)$ is the same, just it is now over this new restricted support. We know from part (a) that $\ell(\theta)$ is maximized at $\hat{\theta} = \bar{Y}$. Since $\ell(\theta)$ is concave down on $[0, 1]$, we know that it is also concave down on $[0, \theta]$. So the restricted MLE of θ is still \bar{Y} unless $\bar{Y} > \theta_0$. So the restricted MLE of θ on $[0, \theta_0]$ is $\hat{\theta}_0 = \min(\bar{Y}, \theta_0)$.

c.

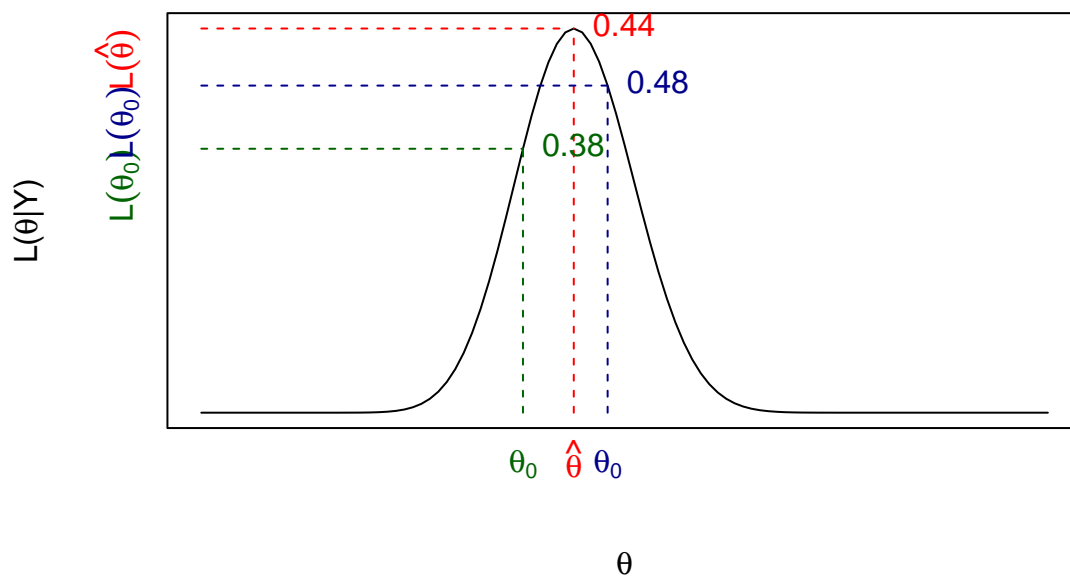
```
plot(theta_values, likelihood_values, type = "l", col = "black",
      xaxt = "n", yaxt = "n", xlab = latex2exp::TeX(r'(\theta)'),
      ylab = latex2exp::TeX(r'(L(\theta|Y))'),
      main = latex2exp::TeX(r'(Likelihood Function for \theta (Bernoulli Sample))'),
      cex.main = 1.3)

# Mark the maximum likelihood estimate (MLE)
mle <- theta_values[which.max(likelihood_values)]
lines(x = rep(mle, times = 44),
      y = seq(0, max(likelihood_values), length.out = 44),
      col = "red", lty = 2)
lines(x = theta_values[which(theta_values < 0.44)],
      y = rep(max(likelihood_values),
              times = length(theta_values[which(theta_values < 0.44)])),
      col = "red", lty = 2)
llh_theta_0 <- ber_likelihood(theta_0, data)
lines(x = theta_values[which(theta_values <= theta_0)],
      y = rep(llh_theta_0, times = length(theta_values[which(theta_values <= theta_0)])),
      col = "darkgreen", lty = 2)
lines(x = rep(theta_0, times = 44),
      y = seq(0, llh_theta_0, length.out = 44),
      col = "darkgreen", lty = 2)
llh_theta_greater <- ber_likelihood(0.48, data)
lines(x = theta_values[which(theta_values <= 0.48)],
      y = rep(llh_theta_greater, times = length(theta_values[which(theta_values <= 0.48)])),
      col = "darkblue", lty = 2)
lines(x = rep(0.48, times = 44),
      y = seq(0, llh_theta_greater, length.out = 44),
      col = "darkblue", lty = 2)
text(mle, max(likelihood_values), round(mle, 2), pos = 4, col = "red")
text(theta_0, llh_theta_0, theta_0, pos = 4, col = "darkgreen")
text(0.48, llh_theta_greater, 0.48, pos = 4, col = "darkblue")
mtext(expression(hat(theta)), side = 1, line = 0.5, col = "red", at = c(mle))
mtext(expression(L(hat(theta))), side = 2, line = 0.5, col = "red", at = max(likelihood_valu
mtext(expression(theta[0]), side = 1, line = 0.5, col = "darkgreen", at = c(theta_0))
```



```
mtext(expression(L(theta[0])), side = 2, line = 0.5, col = "darkgreen", at = llh_theta_0, ad  
mtext(expression(theta[0]), side = 1, line = 0.5, col = "darkblue", at = 0.48)  
mtext(expression(L(theta[0])), side = 2, line = 0.5, col = "darkblue", at = llh_theta_greate
```

Likelihood Function for θ (Bernoulli Sample)



d.

Finding the likelihood test statistic, we take the ratio of the likelihood function under the null hypothesis over the unrestricted likelihood function.

$$\begin{aligned}\lambda(\mathbf{y}) &= \frac{L(\theta_0|\mathbf{y})}{L(\hat{\theta}|\mathbf{y})} \\ &= \frac{(\theta_0)^{\sum_{i=1}^m y_i} (1 - \theta_0)^{m - \sum_{i=1}^m y_i}}{(\hat{\theta})^{\sum_{i=1}^m y_i} (1 - \hat{\theta})^{m - \sum_{i=1}^m y_i}} \\ &= \left[\frac{\theta_0(1 - \hat{\theta})}{\hat{\theta}(1 - \theta_0)} \right]^{\sum_{i=1}^m y_i} \left(\frac{1 - \theta_0}{1 - \hat{\theta}} \right)^m.\end{aligned}$$



Substituting in $\theta_0 = \hat{\theta}_0 = \min(\bar{Y}, \theta_0)$ and $\hat{\theta} = \bar{Y}$,

$$\lambda(\mathbf{y}) = \left[\frac{\min(\bar{y}, \theta_0)(1 - \bar{y})}{\bar{y}(1 - \min(\bar{y}, \theta_0))} \right]^{\sum_{i=1}^m y_i} \left(\frac{1 - \min(\bar{y}, \theta_0)}{1 - \bar{y}} \right)^m$$
$$= \begin{cases} \left[\frac{\theta_0(1 - \bar{y})}{\bar{y}(1 - \theta_0)} \right]^{\sum_{i=1}^m y_i} \left(\frac{1 - \theta_0}{1 - \bar{y}} \right)^m & \theta_0 < \hat{\theta}, \\ 1 & \theta_0 \geq \hat{\theta}. \end{cases}$$

The likelihood ratio test is the test that rejects H_0 for $\{\mathbf{y} : \lambda(\mathbf{y}) \leq c\}$ where $0 \leq c \leq 1$.

e.

```
sum_y <- seq(3.8, 10, 0.1)

sample_size <- c(10)

lambda <- function(n) {
  ((theta_0*(1-(sum_y/n)))/((sum_y/n)*(1- theta_0)))^(sum_y)*((1 - theta_0)/(1-(sum_y/n)))^n
}

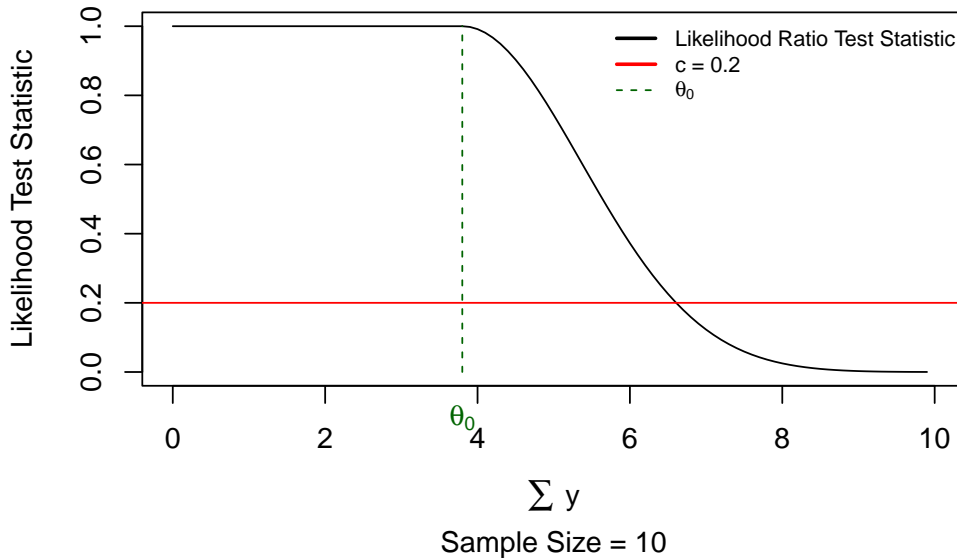
ber_lambda <-lambda(sample_size)

plot(sum_y, ber_lambda, type = "l", col = "black",
      xlab = latex2exp::TeX(r'(\sum y)'),
      ylab = "Likelihood Test Statistic",
      xlim = c(0, 10),
      main = latex2exp::TeX(r'(Likelihood Ratio Test Statistic ($\theta_0$ = 0.38))'),
      cex.main = 1.3)

title(sub = "Sample Size = 10")
lines(seq(0, 3.8, 0.1), rep(1, length(seq(0, 3.8, 0.1))), col = "black")
lines(x = rep(sample_size*theta_0, times = 44),
      y = seq(0, 1, length.out = 44),
      col = "darkgreen", lty = 2)
abline(h = 0.2, col = "red")
mtext(expression(theta[0]), side = 1, line = 0.5, col = "darkgreen", at = c(sample_size*theta_0, 0.2))
legend("topright", c("Likelihood Ratio Test Statistic",
                     "c = 0.2",
                     expression(theta[0])),
      col = c("black", "red", "darkgreen"),
      lwd = c(2, 2, 1),
      lty = c(1, 1, 2),
      bty = "n",
      cex = 0.75)
```



Likelihood Ratio Test Statistic ($\theta_0 = 0.38$)



f.

If $T(\mathbf{Y})$ is a sufficient statistic for θ and $\lambda^*(t)$ and $\lambda(\mathbf{y})$ are LRT statistics based on $T(\mathbf{Y})$ and \mathbf{Y} respectively, $\lambda^*(T(\mathbf{Y})) = \lambda(\mathbf{Y})$ for every $\mathbf{y} \in \mathcal{Y}$. Hence we can show that if $T(\mathbf{Y}) = \bar{Y}$, then

$$\lambda^*(\bar{y}) = \left[\frac{\theta_0(1-\bar{y})}{\bar{y}(1-\theta_0)} \right]^{m\bar{y}} \left(\frac{1-\theta_0}{1-\bar{y}} \right)^m.$$

We can show this function is monotone decreasing by taking the derivative,

$$\begin{aligned} \log(\lambda^*(\bar{y})) &= m\bar{y} \log(\theta_0) + m \log(1-\theta_0) - m\bar{y} \log(1-\theta_0) \\ &\quad - m\bar{y} \log(\bar{y}) - m \log(1-\bar{y}) - m\bar{y} \log(1-\bar{y}) \\ \implies \frac{\partial}{\partial \bar{y}} \log(\lambda^*(\bar{y})) &= m \log(\theta_0) - m \log(1-\theta_0) - m \log(\bar{y}) - \frac{m\bar{y}}{\bar{y}} - \left(\frac{m}{1-\bar{y}} + m \log(1-\bar{y}) + \frac{m\bar{y}}{1-\bar{y}} \right) \\ &= m \log \left(\frac{\theta_0}{(1-\theta_0)\bar{y}} \right) + m - \frac{m(1-\bar{y})}{(1-\bar{y})} - m \log(1-\bar{y}) \\ \implies \exp \left\{ \frac{\partial}{\partial \bar{y}} \log(\lambda^*(\bar{y})) \right\} &= \left(\frac{\theta_0}{\bar{y}(1-\theta_0)(1-\bar{y})} \right)^m = \frac{\partial}{\partial \bar{y}} \lambda^*(\bar{y}). \end{aligned}$$

Full disclosure, I did not take this derivative correctly. It was nasty. Here was my good faith effort. I know it is monotonically decreasing per my plot, but could not show analytically.

Which is a function that is monotonically decreasing over $T(\mathbf{Y}) = \bar{Y}$. So since \bar{Y} is a sufficient statistic for θ , then $\lambda^*(T(\mathbf{Y})) = \lambda(\mathbf{Y})$ for all $\mathbf{y} \in \mathcal{Y}$. So the LRT is the test that rejects H_0 such that $\{\bar{y} : \lambda^*(t) \leq c\}$ where $0 \leq c \leq 1$, which implies that the LRT is the also test that rejects H_0 such that $\{\bar{y} : t \geq \lambda^{-1}(c) = k\}$ (inequality flips because inverse fn.) where $0 \leq k \leq 1$ and $t = \bar{y}$.



2.

We are given $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Pareto}(\nu, \theta)$, with PDF

$$f(x|\nu, \theta) = \frac{\theta\nu^\theta}{x^{\theta+1}}, \quad \nu < x < \infty, \quad \nu > 0, \theta > 0.$$

a.

We use the Factorization Theorem to find a sufficient statistic for (ν, θ) .

$$\begin{aligned} f(\mathbf{x}|\nu, \theta) &= \prod_{i=1}^n \frac{\theta\nu^\theta}{x_i^{\theta+1}} I_{[\nu, \infty)}(x_i) \\ &= \frac{(\theta\nu^\theta)^n}{\prod_{i=1}^n x_i^{\theta+1}} I_{[\nu, \infty)}(\min(x_i)) \\ &= g(T(\mathbf{x})|\nu, \theta)h(\mathbf{x}) \end{aligned}$$

where $g(T(\mathbf{x})|\nu, \theta) = f(\mathbf{x}|\nu, \theta)$, and $h(\mathbf{x}) = 1$. So then by the Factorization Theorem, $T(\mathbf{x}) = (X_{(1)}, \prod_{i=1}^n X_i)$ is a sufficient statistic for (ν, θ) , where $X_{(1)} = \min(X_i)$.

b.

The likelihood function for ν is

$$L(\nu|\theta = \theta, \mathbf{x}) = \frac{(\theta\nu^\theta)^n}{\prod_{i=1}^n x_i^{\theta+1}} I_{[\nu, \infty)}(\min(x_i)),$$

which is a monotone increasing function over ν . Since $0 < \nu < x_{(1)}$, the value that maximizes $L(\nu|\theta = \theta, \mathbf{x})$ with respect to ν is $X_{(1)}$. Hence the MLE for ν , $\hat{\nu} = X_{(1)}$. As for the MLE of θ , the log-likelihood function, $\ell(\nu, \theta)$, for \mathbf{x} is

$$\ell(\nu, \theta) = n \log(\theta) + n\theta \log(\nu) - (\theta + 1) \sum_{i=1}^n \log(x_i)$$

Finding the global maximum, we can set $\frac{\partial}{\partial \theta} \ell(\nu, \theta) = 0$.

$$\begin{aligned} 0 &= \frac{\partial}{\partial \theta} \ell(\nu, \theta) \\ &= \frac{n}{\theta} + n \log(\nu) - \sum_{i=1}^n \log(x_i) \\ \iff \sum_{i=1}^n \log(x_i) &= \frac{n}{\theta} + n \log(\nu) \\ \iff \hat{\theta} &= \frac{n}{\log(\sum_{i=1}^n (x_i) / \hat{\nu}^n)} \\ \iff \hat{\theta} &= \frac{n}{\log(\sum_{i=1}^n (X_i) / X_{(1)}^n)}. \end{aligned}$$

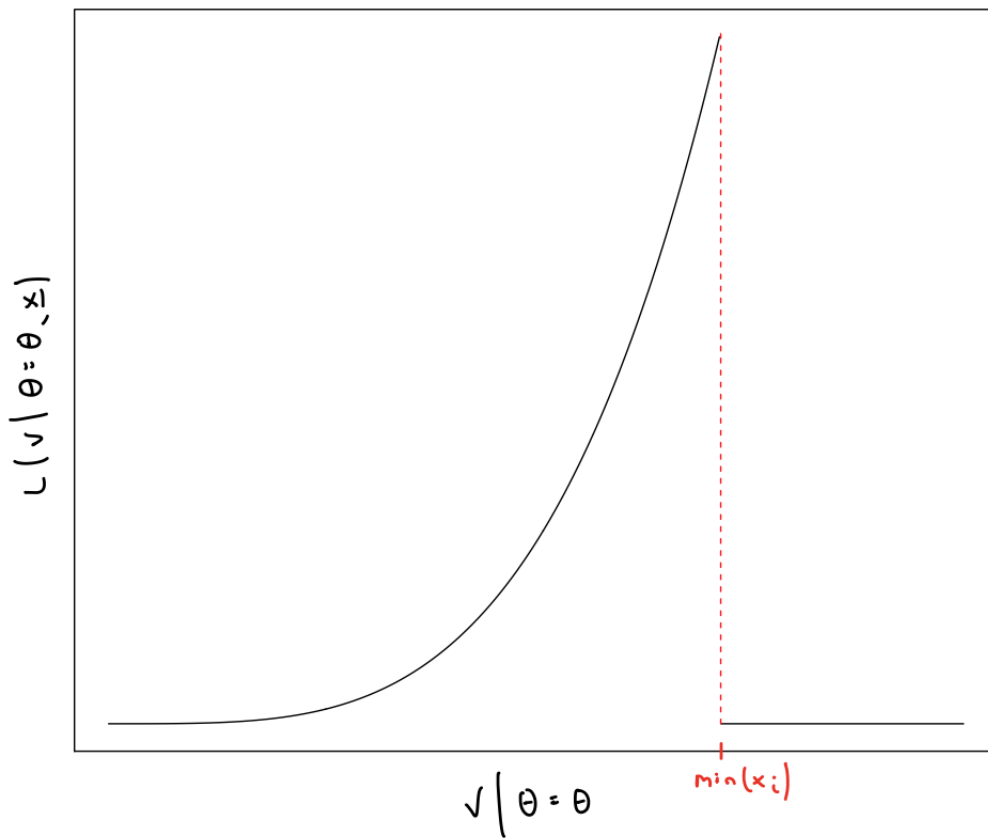


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c.

Since $0 < \nu < x_{(1)}$, the value that maximizes $L(\nu|\theta = \theta, \mathbf{x})$ with respect to ν is $X_{(1)}$. Hence the MLE for ν , $\hat{\nu} = X_{(1)}$.

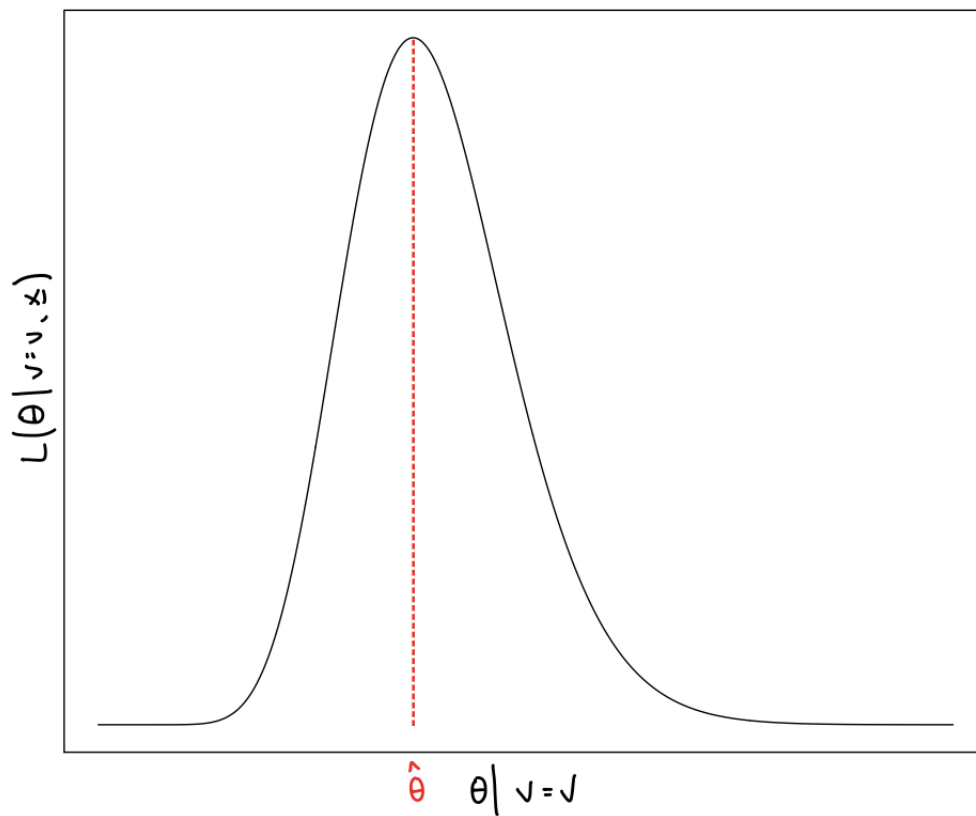
Pareto Likelihood





d.

Pareto Likelihood





e.

With ν unknown, we want to test

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta \neq 1.$$

We know that

$$\hat{\theta} = \frac{n}{\log(\sum_{i=1}^n (X_i)/X_{(1)}^n)}.$$

We write the likelihood ratio test statistic in terms of $T(\mathbf{X}) = \log\left(\frac{\sum_{i=1}^n (X_i)}{X_{(1)}^n}\right) = t$. Finding the likelihood ratio test statistic,

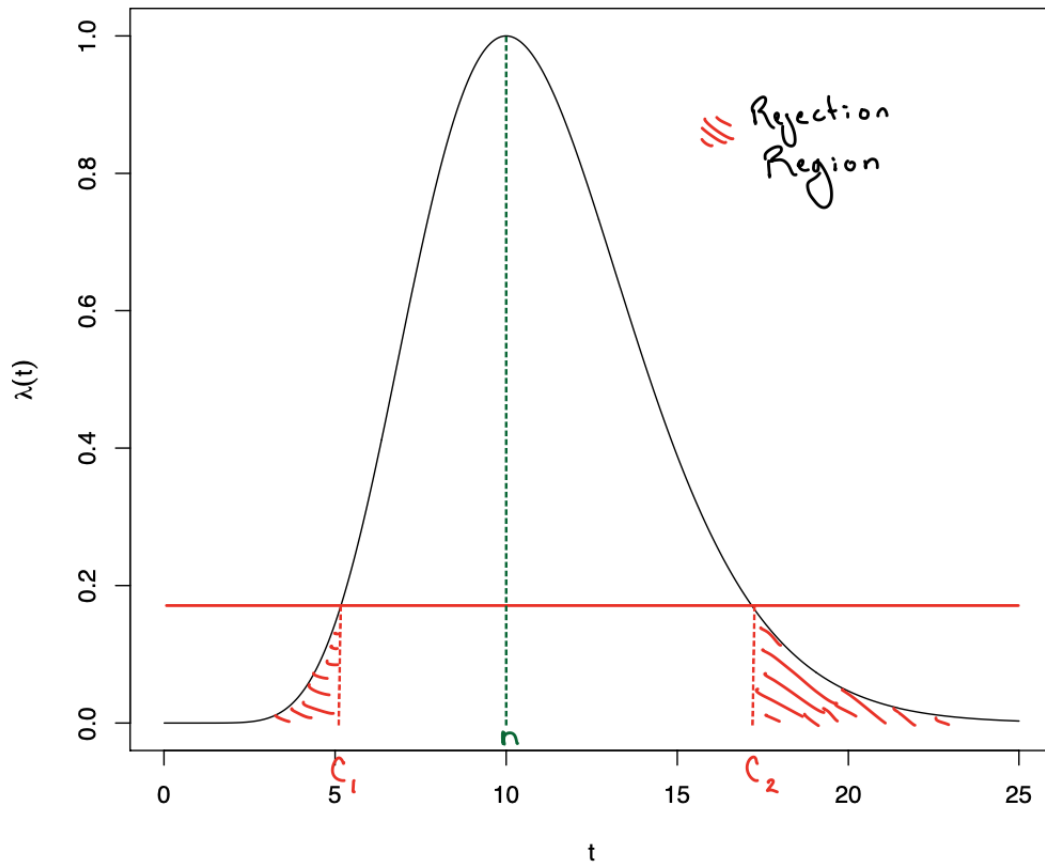
$$\begin{aligned} \lambda(\mathbf{x}) &= \frac{L(1|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})} \\ &= \frac{\hat{\nu}^n (\prod_{i=1}^n x_i)^{\frac{n}{\hat{\nu}}+1}}{\left(\frac{n}{\hat{\nu}}\hat{\nu}\right)^{\frac{n}{\hat{\nu}}} (\prod_{i=1}^n x_i)^2} \\ &= \left(\frac{t}{n}\right)^n \left(x_{(1)}^{(1-\frac{n}{t})}\right)^n \left(\prod_{i=1}^n x_i\right)^{\frac{n}{t}-1} \\ &= \left(\left(\frac{t}{n}\right)^n \left(\frac{x_{(1)}}{\prod_{i=1}^n x_i^{\frac{1}{n}}}\right)^{(1-\frac{n}{t})}\right)^n \\ &= \left(\frac{t}{n}\right)^n \left(\frac{x_{(1)}^n}{\prod_{i=1}^n x_i}\right)^{(1-\frac{n}{t})} \\ &= \left(\frac{t}{n}\right)^n e^{n-t} \quad \left(\text{Since } \left(\frac{x_{(1)}^n}{\prod_{i=1}^n x_i}\right) = e^{-t}\right) \end{aligned}$$

which is a function that is not monotone increasing. Since n is fixed, when $n > t$, the function is decreasing over t , and when $n < t$, the function is increasing over t . Hence the LRT has a critical region of the form $\{\mathbf{x} : T(\mathbf{x}) < c_1 \text{ or } T(\mathbf{x}) > c_2\}$ such that $0 \leq c_1 < c_2 \leq 1$.



f.

Pareto: Likelihood Ratio Test Statistic





3.

We are given $X_1, \dots, X_n \stackrel{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$, where σ^2 is known. We consider testing

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

We know that the likelihood function for an IID normal sample is

$$L(\theta, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \theta)^2}{2\sigma^2}\right),$$

yielding the likelihood ratio test statistic

$$\begin{aligned} \lambda(\mathbf{x}) &= \frac{L(\theta_0, \sigma^2)}{L(\hat{\theta}, \sigma^2)} \\ &= \exp\left\{-\frac{1}{2\sigma^2} \left(\sum_{i=1}^n (x_i - \bar{x})^2 - \sum_{i=1}^n (x_i - \theta_0)^2\right)\right\} \\ &= \exp\left\{-\frac{1}{2\sigma^2} \left(-\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \theta_0)^2 + \sum_{i=1}^n (x_i - \bar{x})\right)\right\} \\ &= \exp\left\{\left(-\frac{1}{2\sigma^2}\right) (+n(\bar{x} - \theta_0)^2)\right\} \\ &= \exp\left\{\frac{-n(\bar{x} - \theta_0)^2}{2\sigma^2}\right\} \\ \implies \lambda(\mathbf{x}) &= \begin{cases} \exp\left\{\frac{-n(\bar{x} - \theta_0)^2}{2\sigma^2}\right\}, & \bar{x} < \theta_0 \\ 1, & \bar{x} \geq \theta_0. \end{cases} \end{aligned}$$

The likelihood ratio test is the test that rejects H_0 for $\{\mathbf{x} : \lambda(\mathbf{x}) \leq c\}$ where $0 \leq c \leq 1$.

```
xbar <- seq(-2, 2, length.out = 100)

# function computes LRT
lrt <- function(xbar, theta_0, sigma_sq, n) {
  output <- exp(-n*(xbar - theta_0)^2/(2*sigma_sq))
  for (i in seq_along(xbar)) {
    if (xbar[i] >= theta_0) output[i] <- 1
  }
  output
}

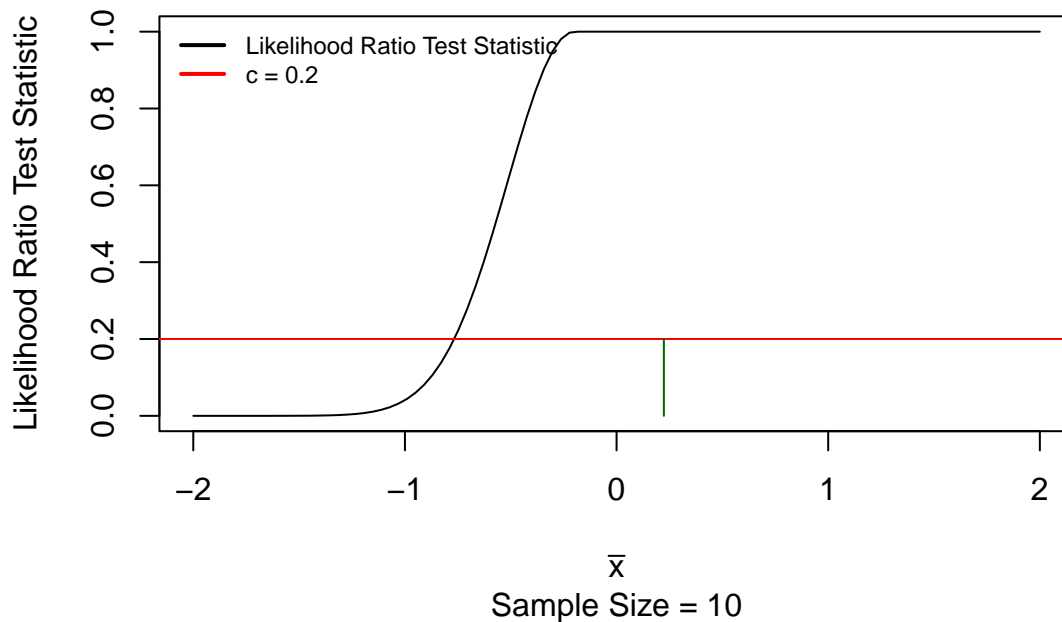
lrt <- lrt(xbar, theta_0 = -0.2, sigma_sq = 1, n = 10)

# plot
plot(xbar, lrt, type = "l", col = "black",
      xlab = latex2exp::TeX(r'(\bar{x})'),
```



```
ylab = "Likelihood Ratio Test Statistic",  
xlim = c(-2, 2),  
main = latex2exp::TeX(r'(Likelihood Ratio Test Statistic ( $\theta_0 = -0.2$ ))),  
cex.main = 1.3)  
title(sub = "Sample Size = 10")  
abline(h = 0.2, col = "red")  
lines(x = rep(lrt[32], length = 10), y = seq(0, 0.2, length.out = 10), col = "darkgreen")  
legend("topleft", c("Likelihood Ratio Test Statistic",  
                    "c = 0.2"),  
      col = c("black", "red"),  
      lwd = c(2, 2),  
      lty = c(1, 1),  
      bty = "n",  
      cex = 0.75)
```

Likelihood Ratio Test Statistic ($\theta_0 = -0.2$)



The rejection region is any region where $\lambda(\mathbf{x}) \leq c$, where $c = 0.2$ is denoted by the red line in the figure.



4.

a.

We are given $Y \sim \text{IG}(\alpha, \beta)$. Let $W = g(Y) = \frac{1}{Y}$. We then show that $W \sim \text{Gamma}(\alpha, \beta)$.

$$\begin{aligned} f_W w &= f_Y \frac{1}{w} \left| \frac{d}{dw} \frac{1}{w} \right|, \quad 0 < \frac{1}{w} < \infty \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(w^{-1})^{\alpha+1}} \exp\left\{-\frac{1}{\beta(w^{-1})}\right\}, \quad 0 < w < \infty \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} w^{\alpha+1} \exp\left\{-\frac{w}{\beta}\right\}, \quad 0 < w < \infty \\ \implies W &\sim \text{Gamma}(\alpha, \beta). \end{aligned}$$

b.

This is the same problem as Homework 2 problem 6(a).

We have that $S^2|\sigma^2 \sim \chi_{n-1}^2$ with sample size n . This is equivalent to saying that $S^2|\sigma^2 \sim \text{Gamma}\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$. We suppose the prior distribution $\pi(\sigma^2)$ is an inverse gamma with shape parameter α and scale parameter β ,

$$\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left\{-\frac{1}{\beta\sigma^2}\right\}, \quad 0 < \sigma^2 < \infty.$$

We know that the posterior distribution for $\sigma^2|S^2$ is

$$\pi(\sigma^2|S^2) = \frac{f(S^2|\sigma^2)\pi(\sigma^2)}{m(S^2)}.$$

Knowing $f(S^2|\sigma^2)$ and $\pi(\sigma^2)$, we can say that

$$\begin{aligned} \pi(\sigma^2|S^2) &\propto \frac{1}{(\sigma^2)^{(n-1)/2}} \exp\left\{-\frac{(n-1)S^2}{2\sigma^2}\right\} \frac{1}{(\sigma^2)^{\alpha+1}} \exp\left\{-\frac{1}{\beta\sigma^2}\right\} \\ &= \frac{1}{(\sigma^2)^{(n-1)/2+\alpha+1}} \exp\left\{-\left(\frac{(n-1)S^2}{2\sigma^2} + \frac{1}{\beta\sigma^2}\right)\right\} \\ &= \text{IG}\left(\alpha + \frac{(n-1)}{2}, \left\{\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right\}^{-1}\right). \end{aligned}$$

This implies that the kernel of the posterior is proportional to an $\text{IG}\left(\alpha + \frac{(n-1)}{2}, \left\{\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right\}^{-1}\right)$.

So $\sigma^2|S^2 \sim \text{IG}\left(\alpha + \frac{(n-1)}{2}, \left\{\frac{(n-1)S^2}{2} + \frac{1}{\beta}\right\}^{-1}\right)$.

c.

Here we consider

$$H_0 : \sigma \leq 1 \quad \text{versus} \quad H_1 : \sigma > 1.$$



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To find the region of the sample space for which a Bayes test will decide $\sigma \leq 1$, we consider a transformation of the posterior $\pi(\sigma^2|S^2)$,

$$W = 2(\sigma^2)^{-1} \left(\frac{(n-1)S^2}{2} + \frac{1}{\beta} \right).$$

Note the Jacobean for this transformation is $|\mathbf{J}| = 2 \left(\frac{(n-1)S^2}{2} + \frac{1}{\beta} \right)^{-2}$. (See following pages)