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Course: Theory of Statistics I
Instructor: Dr. Stamey

Theory 1 Quiz 2

1.

We are given $X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2)$. We are interested in $E[S]$ such that $S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$. We begin by adding a 'well chosen' 1,

$$\begin{aligned}
E[S] = E[\sqrt{S^2}] &= \sqrt{\frac{\sigma^2}{(n-1)}} E\left[\sqrt{\frac{S^2(n-1)}{\sigma^2}}\right] \\
&= \sqrt{\frac{\sigma^2}{(n-1)}} \int_0^\infty \sqrt{x} \frac{1}{2^{\frac{\nu-1}{2}} \Gamma\left(\frac{\nu-1}{2}\right)} x^{\frac{\nu-1}{2}-1} e^{-\frac{x}{2}}\, dx \\
&= \sqrt{\frac{\sigma^2}{(n-1)}} \int_0^\infty \frac{1}{2^{\frac{\nu-1}{2}} \Gamma\left(\frac{\nu-1}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}\, dx \\
&= \sqrt{\frac{\sigma^2}{(n-1)}} \frac{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)}{2^{\frac{\nu-1}{2}} \Gamma\left(\frac{\nu-1}{2}\right)} \underbrace{\int_0^\infty \frac{1}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}\, dx}_{\chi^2_\nu \text{ density}} \\
&= \sigma \sqrt{\frac{2}{(n-1)}} \cdot \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)}.
\end{aligned}$$

2.

We are given $X_1 \sim \text{Poisson}(\lambda_1)$, where $Y = X_1 + X_2$ and $Y \sim \text{Poisson}(\lambda)$. We want to find the distribution of X_2.

We know that $Y = X_1 + X_2 \iff X_2 = Y - X_1$. We also know that X_1 and X_2 are independent. By definitions of MGFs and expectations of independent random variables, this would imply that the moment generating function of X_2 is,

$$\begin{aligned}
Me^{tX_2} &= M(e^{t(Y-X_1)}) \\
&= \frac{M_Y(e^{tY})}{M_{X_1}(e^{tX_1})} \\
&= \frac{e^{\lambda(e^t-1)}}{e^{\lambda_1(e^t-1)}} \\
&= e^{\lambda(e^t-1)-\lambda_1(e^t-1)} \\
&= e^{(\lambda-\lambda_1)(e^t-1)}.
\end{aligned}$$

Therefore $X_2 \sim \text{Poisson}(\lambda - \lambda_1)$.

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3.

Given independent random variables $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, and also $U = aX = b$, $V = cY + d$ for constants a, b, c, and d, we can find the MGF of $W = U + V$ by first finding the MGFs for U and V,

$$M_U(t) = M_{aX+b}(t) = e^{bt} M_X(at)$$
$$M_V(t) = M_{cY+d}(t) = e^{dt} M_Y(ct).$$

Then we can find the MGF for $W = U + V$ using definitions of MGFs and expectations of independent random variables,

$$\begin{aligned}
M_W(t) &= M_{U+V}(t) \\
&= E[e^{t(U+V)}] \\
&= E[e^{tU}]E[e^{tV}] \\
&= M_U(t)M_V(t) \\
&= \exp\left(t(b+d)\right) M_X(at)M_Y(ct) \\
&= \exp\left(t(b+d)\right) \exp\left(\mu_X at + \frac{\sigma_X^2 t^2}{2}\right) \exp\left(\mu_Y ct + \frac{\sigma_Y^2 t^2}{2}\right) \\
&= \exp\left(t(b + d + a\mu_X + c\mu_y) + \frac{t^2(\sigma_X^2 + \sigma_Y^2)}{2}\right).
\end{aligned}$$

4.

Given independent random variables $X_1 \sim \text{Unif}(a, b)$ and $X_2 \sim \text{Unif}(a, b)$, and given $Y_1 = X_1 + X_2$, $Y_2 = X_2$, we apply the methods of transformations to acquire **(a)** the joint density of Y_1 and Y_2. We find $X_1 = g_1(y_1, y_2) = Y_1 - Y_2$ and $X_2 = g_2(y_1, y_2) = Y_2$. The Jacobian would be

$$\mathbf{J} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

We know the joint density of X_1 and X_2 is

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{(b-a)^2}, \quad a \le x_1 \le b, \quad a \le x_2 \le b.$$

Therefore, the joint density of Y_1 and Y_2 is

$$\begin{aligned}
f_{Y_1Y_2}(y_1, y_2) &= f_{X_1X_2}(g_1(y_1, y_2), g_2(y_1, y_2))|\mathbf{J}| \\
&= \frac{1}{(b-a)^2}, \quad a \le y_1 - y_2 \le b, \quad a \le y_2 \le b.
\end{aligned}$$

Supposing $a = 0$ and $b = 1$, then $X_1 \sim \text{Unif}(0, 1)$ and $X_2 \sim \text{Unif}(0, 1)$. Then we know the joint density of Y_1 and Y_2 is

$$f_{Y_1Y_2}(y_1, y_2) = 1, \quad 0 \le y_1 - y_2 \le 1, \quad 0 \le y_2 \le 1.$$

(b) Continuing in the case, where $a = 0$ and $b = 1$, we expect the support of Y_1 to be $\mathcal{S}_{Y_1} = [0, 2]$, as the transformation should not be uniform, but is the sum of two values ranging from 0 to 1.

Also the support of Y_2 is $\mathcal{S}_{Y_2} = [0,1]$. Using the convolution formula and rules of independent random variables, we derive the distribution of Y_1

$$f_{Y_1}(y_1) = \int_{y_2 \in \mathcal{S}_{Y_2}} f_{X_1}(y_1 - y_2) f_{X_2}(y_2)\, dy_2 = \int_0^1 f_{X_1}(y_1 - y_2) f_{X_2}(y_2)\, dy_2 = \int_0^1 f_{X_1}(y_1 - y_2)\, dy_2,$$

as $f_{X_2}(y_2) = 1$ for all $0 \le y_2 \le 1$. Additionally, there exists the conditions where if $f_{X_1}(y_1 - y_2) \ge 0$, then $0 \le y_1 - y_2 \le 1$.

- If $y_1 \in [0,1]$, we already have $y_1 - y_2 \le 1$ since $y_1 \le 1$ (and $y_2 \in [0,1]$). We also need $y_1 - y_2 \ge 0$ for the density to be nonzero; $y_2 \le y_1$. Then our integral becomes:

$$\begin{aligned} f_{y_1}(y_1) &= \int_0^{y_1} f_{X_1}(y_1 - y_2)\, dy_2 + \int_{y_1}^1 f_{X_1}(y_1 - y_2)\, dy_2 \\ &= \int_0^{y_1} 1\, dy_2 + 0 \quad \text{(since } y_2 \le y_1) \\ &= [y_2]_0^{y_1} = y_1 \end{aligned}$$

- If $y_1 \in [1,2]$, we already have $y_1 - y_2 \ge 0$ since $y_1 \ge 1$ (and $y_2 \in [0,1]$). We then need $y_1 - y_2 \le 1$ for the density to be nonzero; $y_2 \ge y_1 - 1$. Hence, our integral becomes:

$$\begin{aligned} f_{y_1}(y_1) &= \int_0^{y_1 - 1} f_{X_1}(y_1 - y_2)\, dy_2 + \int_{y_1 - 1}^1 f_{X_1}(y_1 - y_2)\, dy_2 \\ &= 0 + \int_{y_1 - 1}^1 1\, dy_2 \quad \text{(since } y_2 \ge y_1 - 1) \\ &= [y_2]_{y_1 - 1}^1 = 2 - y_1. \end{aligned}$$

With this in mind, we know that the marginal PDF of $Y_1 = X_1 + X_2$ is

$$f_{Y_1} = \begin{cases} y_1, & 0 \le y_1 \le 1 \\ 2 - y_1, & 1 \le y_1 \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

5.

We are given $X|\tau \sim \mathcal{N}(0, \sigma^2 = \frac{1}{\tau})$, where $\tau \sim \text{Gamma}(\alpha, \lambda)$. **(a)** We can find the marginal distribution of X by summing over the support of τ for $f_{X,\tau}(x,\tau) = f_\tau(\tau)f(x|\tau)$.

$$\begin{aligned}
\int_0^\infty f_{X,\tau}(x,\tau) &= \int_0^\infty f(\tau)f(x|\tau)\,d\tau \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)}\tau^{\alpha-1}\exp\left(-\lambda\tau\right)\frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{x^2\tau}{2}\right)\,d\tau \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)\sqrt{2\pi}}\int_0^\infty \tau^{(\alpha-\frac{1}{2})-1}\exp\left(-\left(\frac{x^2}{2}+\lambda\right)\tau\right)\,d\tau \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)\sqrt{2\pi}}\frac{\Gamma\left(\alpha-\frac{1}{2}\right)}{\left(\frac{x^2}{2}+\lambda\right)^{\alpha-\frac{1}{2}}}\underbrace{\int_0^\infty \frac{\left(\frac{x^2}{2}+\lambda\right)^{\alpha-\frac{1}{2}}}{\Gamma\left(\alpha-\frac{1}{2}\right)}\tau^{(\alpha-\frac{1}{2})-1}\exp\left(-\left(\frac{x^2}{2}+\lambda\right)\tau\right)\,d\tau}_{\text{Gamma}\left(\alpha-\frac{1}{2},\frac{x^2}{2}+\lambda\right)\text{ density}} \\
\implies f_X(x) &= \frac{\lambda^\alpha}{\Gamma(\alpha)\sqrt{2\pi}}\frac{\Gamma\left(\alpha-\frac{1}{2}\right)}{\left(\frac{x^2}{2}+\lambda\right)^{\alpha-\frac{1}{2}}}.
\end{aligned}$$

(b) Finding the conditional distribution of $\tau|X$, we can divide the joint distribution $f_{X,\tau}(x,\tau) = f_\tau(\tau)f(x|\tau)$ by the marginal distribution of X.

$$\begin{aligned}
f(\tau|x) &= \frac{\frac{\lambda^\alpha}{\Gamma(\alpha)\sqrt{2\pi}}\tau^{(\alpha-\frac{1}{2})-1}\exp\left(-\left(\frac{x^2}{2}+\lambda\right)\tau\right)}{\frac{\lambda^\alpha}{\Gamma(\alpha)\sqrt{2\pi}}\frac{\Gamma\left(\alpha-\frac{1}{2}\right)}{\left(\frac{x^2}{2}+\lambda\right)^{\alpha-\frac{1}{2}}}} \\
&= \frac{\left(\frac{x^2}{2}+\lambda\right)^{\alpha-\frac{1}{2}}}{\Gamma\left(\alpha-\frac{1}{2}\right)}\tau^{(\alpha-\frac{1}{2})-1}\exp\left(-\left(\frac{x^2}{2}+\lambda\right)\tau\right) \\
\implies \tau|X &\sim \text{Gamma}\left(\alpha-\frac{1}{2},\frac{x^2}{2}+\lambda\right).
\end{aligned}$$

6.

We are given $X_1 \sim \chi^2_{\nu_1}$ and $X_2 \sim \chi^2_{\nu_2}$, and they are assumed to be independent. We define $Y_1 = \frac{X_1}{\nu_1} / \frac{X_2}{\nu_2}$ and $Y_2 = X_2$. We will first find the joint distribution of Y_1 and Y_2. This would imply that $X_1 = \frac{\nu_1}{\nu_2} Y_1 Y_2$ and $X_2 = Y_2$. We can find the Jacobian, $\mathbf{J}$, as

$$\mathbf{J} = \begin{vmatrix} \frac{\nu_1}{\nu_2} y_2 & \frac{\nu_1}{\nu_2} y_1 \\ 0 & 1 \end{vmatrix} = \frac{\nu_1}{\nu_2} y_2.$$

We know that the joint distribution of X_1 and X_2 is

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} x_1^{\frac{\nu_1}{2}-1} x_2^{\frac{\nu_2}{2}-1} e^{-\frac{(x_1+x_2)}{2}}, \quad x_1 > 0, \quad x_2 > 0.$$

By the method of transformations, we have,

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \frac{1}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2} y_1 y_2\right)^{\frac{\nu_1}{2}-1} y_2^{\frac{\nu_2}{2}-1} e^{-\frac{y_2}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)} \left(\frac{\nu_1}{\nu_2} y_2\right), \quad y_1 > 0, \quad y_2 > 0, \\ &= \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y_1^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} y_2^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{y_2}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)}, \quad y_1 > 0, \quad y_2 > 0, \end{aligned}$$

as the join distribution of Y_1 and Y_2. Finding the marginal pdf of Y_1, we need to integrate over the support of Y_2.

$$f_{Y_1}(y_1) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y_1^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} \int_0^\infty y_2^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{y_2}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)} dy_2.$$

We recognize that within the integral is the kernel of a Gamma distribution with shape $\alpha = \frac{\nu_1+\nu_2}{2}$ and rate $\lambda = \frac{1}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)$. Adding a 'well chosen' 1, we have,

$$f_{Y_1}(y_1) = \frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y_1^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\left[\frac{1}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)\right]^{\frac{\nu_1+\nu_2}{2}}} \underbrace{\int_0^\infty \frac{\left[\frac{1}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)\right]^{\frac{\nu_1+\nu_2}{2}}}{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)} y_2^{\frac{\nu_1+\nu_2}{2}-1} e^{-\frac{y_2}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)} dy_2}_{\text{Gamma}\left(\frac{\nu_1+\nu_2}{2}, \frac{1}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)\right) \text{ density}},$$

giving us,

$$\frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y_1^{\frac{\nu_1}{2}-1}}{2^{\frac{\nu_1+\nu_2}{2}} \Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\left[\frac{1}{2}\left(\frac{\nu_1}{\nu_2} y_1 + 1\right)\right]^{\frac{\nu_1+\nu_2}{2}}} = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} y_1^{\frac{\nu_1}{2}-1} \left(\frac{\nu_1}{\nu_2} y_1 + 1\right)^{-\frac{\nu_1+\nu_2}{2}}$$

such that $y_1 > 0$, where $Y_1 \sim \mathbf{F}(\nu_1, \nu_2)$.

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7.

We are given random variables $W \sim \mathcal{N}(0,1)$, and $V \sim \chi^2_r$. We are are interested in the distribution of T^2 where $T = W/\sqrt{\frac{V}{r}}$. So then $T^2 = W^2/\frac{V}{r}$. We know that any random variable distributed $\mathcal{N}(0,1)$ that is squared has a χ^2_1 distribution. So then T^2 is the ratio of two χ^2 random variables divided by their respective degrees of freedom ($T^2 = \frac{W^2}{1}/\frac{V}{r}$). Based off of the result in problem (6) we know then that $T^2 \sim \mathbf{F}(1,r)$.