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Course: Theory of Statistics I
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Theory 1 Quiz 1

1.

a.

To verify $f(x)$ as a pdf, we can check if its cdf over its whole support is equal to one.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1+\alpha}{x^{(2+\alpha)}}\,dx &= \int_{1}^{\infty} \frac{1+\alpha}{x^{(2+\alpha)}}\,dx \\ &= (1+\alpha)\int_{1}^{\infty} \frac{1}{x^{(2+\alpha)}}\,dx \\ &= (1+\alpha)\left[-\frac{1}{(1+\alpha)x^{(1+\alpha)}}\Big|_{1}^{\infty}\right] \\ &= \left[-\frac{1}{x^{(1+\alpha)}}\Big|_{1}^{\infty}\right] \\ &= 0-(-1) \\ &= 1 \quad \square\end{aligned}$$

So $f(x)$ is a valid pdf.

b.

Let c be greater than 1. So then

$$\begin{aligned}P(X>c) &= \int_{c}^{\infty} \frac{1+\alpha}{x^{(2+\alpha)}}\,dx \\ &= (1+\alpha)\int_{c}^{\infty} \frac{1}{x^{(2+\alpha)}}\,dx \\ &= (1+\alpha)\left[-\frac{1}{(1+\alpha)x^{(1+\alpha)}}\Big|_{c}^{\infty}\right] \\ &= \left[-\frac{1}{x^{(1+\alpha)}}\Big|_{c}^{\infty}\right] \\ &= 0-(-c^{(1+\alpha)}) \\ &= c^{(1+\alpha)} \quad \square\end{aligned}$$

2.

a.

Let D represent the event a widget is defective. We are interested in $P(D = 3)$. So then,

$$\begin{aligned} P(D = 3) &= \frac{\binom{4}{3}\binom{16}{0}}{\binom{20}{3}} \\ &= \frac{4(1)}{1140} \\ &\approx 0.00351. \end{aligned}$$

b.

We are interested in $P(D \geq 1)$. So then,

$$\begin{aligned} P(D \geq 1) &= 1 - P(D = 0) \\ &= 1 - \frac{\binom{4}{0}\binom{16}{3}}{\binom{20}{3}} \\ &= 1 - \frac{560}{1140} \\ &= 1 - 0.3\bar{8} \\ &\approx 0.611. \end{aligned}$$

c.

We are interested in $P(D^c = 3 | D^c = 2)$, which represents drawing an additional non-defective unit given two have been drawn. So then by the definition of Conditional Probability,

$$\begin{aligned} P(D^c = 3 | D^c = 2) &= \frac{P(D^c = 3 \cap D^c = 2)}{P(D^c = 2)} \\ &= \frac{P(D^c = 3)}{P(D^c = 2)} \\ &= \frac{\left(\frac{\binom{4}{0}\binom{16}{3}}{\binom{20}{3}}\right)}{\left(\frac{\binom{4}{0}\binom{16}{2}}{\binom{20}{2}}\right)} \\ &= \frac{\left(\frac{560}{1440}\right)}{\left(\frac{120}{190}\right)} \\ &\approx 0.6157. \end{aligned}$$

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3.

a.

Let M be the event that a student is a male, and let F be the event that a student attends football games. We are interested in $P(F)$. By the Total Law of Probability, we have

$$\begin{aligned}
P(F) &= P(F \cap M)P(M) + P(F \cap M^c)P(M^c) \\
&= (0.25)(0.45) + (0.20)(0.55) \\
&= 0.2225.
\end{aligned}$$

b.

$$\begin{aligned}
P(M|F) &= \frac{P(F \cap M)P(M)}{P(F)} \\
&= \frac{(0.25)(0.45)}{0.2225} \\
&\approx 0.5056.
\end{aligned}$$

4.

We have that $f(x) = 2x$, where $0 < x < 1$. Finding the pdf of $Y = \sqrt{X}$, we can first find

$$\begin{aligned}
P(Y \leq y) &= P(\sqrt{X} \leq y) \\
&= P(X \leq y^2) \\
&= F_X(y^2) \\
\implies f_Y(y) &= \frac{d}{dy} F_X(y^2) \\
&= f_X(y^2)\left|\frac{d}{dy} y^2\right| \\
&= 2y^2(2y) \\
&= 4y^3, \qquad \sqrt{0} < y < \sqrt{1}.
\end{aligned}$$

So then we have

$$f_Y(y) = \begin{cases} 4y^3, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

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5.

Given $X \sim \text{Geo}(p)$ where X is the number of failures until the first success, we have $f(x) = (1-p)^x p$, $x = 0, 1, 2...$ We are looking for the pmf of $Y = X + 1$.

$$
\begin{aligned}
P(Y = y) &= P(X + 1 = y) \\
&= P(X = y - 1) \\
\implies f_X(y-1) &= (1-p)^{y-1} p \\
&= f_Y(y)
\end{aligned}
$$

So then we have

$$
f_Y(y) = (1-p)^{y-1} p, \qquad y = 1, 2, 3...
$$

6.

a.

To find the value of c that validates this pdf, we can solve for c,

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} c(1-x)^2 \, dx \\
&= \int_0^1 c(1-x)^2 \, dx \\
&= c \int_0^1 (1-x)^2 \, dx \\
&= c \left(\int_0^1 x^2 \, dx - 2 \int_0^1 x \, dx + \int_0^1 1 \, dx \right) \\
&= c \left(\frac{1}{3} x^3 - x^2 + x \Big|_0^1 \right) \\
&= c \left(\frac{1}{3} - 1 + 1 \right) \\
\implies c &= 3.
\end{aligned}
$$

b.

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} (x) 3(1-x)^2 \, dx \\
&= 3 \int_0^1 (x)(1-x)^2 \, dx \\
&= 3 \left(\int_0^1 x^3 \, dx - 2 \int_0^1 x^2 \, dx + \int_0^1 x \, dx \right) \\
&= 3 \left(\frac{1}{4} x^4 - \frac{2}{3} x^3 + \frac{1}{2} x^2 \Big|_0^1 \right) \\
&= 3 \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) \\
&= 0.25.
\end{aligned}
$$

We know $E[X]^2$, but we need to find $E[X^2]$ to find $\text{Var}[X]$.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} (x^2)3(1-x)^2 \, dx \\ &= 3\int_0^1 (x^2)(1-x)^2 \, dx \\ &= 3\left(\int_0^1 x^4 \, dx - 2\int_0^1 x^3 \, dx + \int_0^1 x^2 \, dx\right) \\ &= 3\left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\Big|_0^1\right) \\ &= 3\left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) \\ &= 0.1. \end{aligned}$$

So then

$$\begin{aligned} \text{Var}[X] &= E[X^2] - E[X]^2 \\ &= 0.1 - (0.25)^2 \\ &= 0.0375. \end{aligned}$$

c.

We know $\text{Var}[X] = 0.0375 \iff \sigma = 0.1936$, and $\mu = 0.25$. So then $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-0.1373 < X < 0.6373) = P(0 < X < 0.6373)$. This gives us

$$\begin{aligned} P(0 < X < 0.6373) &= 3\int_0^{0.6373} (1-x)^2 \, dx \\ &= 3\left(\frac{1}{3}x^3 - x^2 + x\Big|_0^{0.6373}\right) \\ &\approx 0.9523. \end{aligned}$$

7.

a.

Let D be the event that someone possesses the disease, hence $P(D) = 0.01$. For test A, $P(A = +|D) = 0.90$ and $P(A = -|D^c) = 0.94$. For test B, $P(B = +|D) = 0.95$ and $P(A = -|D^c) = 0.92$. We are interested in $P(D|A \cap B = +)$. By Bayes' Rule, and assuming the tests are independent, we know that

$$\begin{aligned} P(D|A \cap B = +) &= \frac{P(A = +|D)P(B = +|D)P(D)}{P(A = +|D)P(B = +|D)P(D) + P(A = +|D^c)P(B = +|D^c)P(D^c)} \\ &= \frac{(0.90)(0.95)(0.01)}{(0.90)(0.95)(0.01) + (1 - 0.94)(1 - 0.92)(0.99)} \\ &\approx 0.6428. \end{aligned}$$

b.

Now we are interested in $P(B = +|A = +)$. Since A and B are independent tests, then by the Total Law of Probability,

$$\begin{aligned} P(B = +|A = +) &= P(B = +) \\ &= P(B = +|D)P(D) + P(B = +|D^c)P(D^c) \\ &= (0.95)(0.01) + (1 - 0.92)(0.99) \\ &= 0.0887. \end{aligned}$$

8.

We have $\psi(t) = \ln(M_X(t))$ is a moment generating function. So $\psi(t) = \ln(E[e^{tX}])$. Taking the first derivative of the function, we have it that

$$\begin{aligned} \psi'(0) &= \frac{d}{dt}(\psi(0)) \\ &= \frac{d}{dt}\left(\ln(E[e^{tX}])\right)\Big|_{t=0} \\ &= \frac{E[Xe^{tX}]}{E[e^{tX}]}\Big|_{t=0} \\ &= E[X] \\ &= \mu. \quad \square \end{aligned}$$

Also, we can see that

$$\begin{aligned} \psi''(0) &= \frac{d^2}{d^2t}(\psi(0)) \\ &= \frac{d^2}{d^2t}\left(\ln(E[e^{tX}])\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{E[Xe^{tX}]}{E[e^{tX}]}\right)\Big|_{t=0} \\ &= \frac{E[X^2e^{tX}]E[x^{tX}] - E[Xe^{tX}]^2}{E[e^{tX}]^2}\Big|_{t=0} \qquad \text{(by quotient rule)} \\ &= E[X^2] - E[X]^2 \\ &= \sigma^2. \quad \square \end{aligned}$$