# STA 5364, Report 1.12

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### Report 1.12

Suppose $X \sim \text{Exponential}(\lambda)$ is independent of $Y \sim \text{Exponential}(\eta)$. Use these distributions to illustrate each of the results in (1.3.1) - (1.3.10). Note that in the case of the sums, the result is the so called Erlang distribution.

*1.3.1*

Consider $X$, which is a positive random variable with survival function $S(x) = e^{-\lambda x}$. Let $Y = 1/X$ be a transformation of the random variable $X$. Using the method of transformations, we have that

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad X = \frac{1}{Y}, \quad \frac{dx}{dy} = -\frac{1}{y^2} \implies \left|\frac{dx}{dy}\right| = \frac{1}{y^2}.$$

So then we have,

$$\begin{aligned} f_Y(y) &= \frac{\lambda}{y^2} e^{-\frac{\lambda}{y}}, \quad y > 0 \\ \implies G_Y(y) &= \int_0^y \frac{\lambda}{t^2} e^{-\frac{\lambda}{t}}\, dt \\ &= \int_\infty^{\frac{\lambda}{y}} e^{-u}\, du, \quad \left(u = \frac{\lambda}{t},\ du = -\frac{\lambda}{t^2} dt\right) \\ &= e^{-\frac{\lambda}{y}}, \quad y > 0. \end{aligned}$$

So now having $G_Y(y)$, $S(y) = 1 - G_Y(y) = 1 - e^{-\frac{\lambda}{y}} \implies S(1/y) = 1 - G_X(y) = 1 - (1 - e^{-\lambda y}) = G_Y(y)$.

*1.3.2*

We are aiming use the case of two exponential competing risks $X$, $Y$, and their minimum $\min\{X,\ Y\}$, with CDF's $F$, $G$, and $U$ respectively, that for $w \in \mathbb{R}$,

$$\bar{U}(w) = \bar{F}(w)\bar{G}(w).$$

To find the joint survival function $\bar{U}(w)$, we employ the survival functions of $X$ and $Y$, with $w \in \mathbb{R}$,

$$\bar{U}(w) = \bar{F}_X(w)\bar{G}_Y(w) = e^{-\lambda w} e^{-\eta w} = e^{-w(\lambda + \eta)}, \quad w > 0.$$

This yields the survival function for $\min\{X,\ Y\}$.

*1.3.3*

We can also use the distributions of $X$ and $Y$ to illustrate the density of $\min\{X,\ Y\}$. Consider the survival function and its relationship to the CDF of $\min\{X,\ Y\}$. So we have that $\bar{U}(w) = e^{-w(\lambda+\eta)} \implies U(w) = 1 - e^{-w(\lambda+\eta)}$. Then,

$$\begin{aligned}u(w) &= \frac{d}{dw}1 - e^{-w(\lambda+\eta)} \\ &= (\lambda+\eta)e^{-w(\lambda+\eta)} \\ &= e^{-\eta w}\lambda e^{-\lambda w} + e^{-\lambda w}\eta e^{-\eta w} \\ &= f(w)\bar{G}(w) + g(w)\bar{F}(w).\end{aligned}$$

*1.3.4*

Finding the hazard rate for $U = \min\{X,\ Y\}$, we have that

$$h_U(w) = \frac{u(w)}{\bar{U}(w)} = \frac{f(w)\bar{G}(w) + g(w)\bar{F}(w)}{\bar{F}(w)\bar{G}(w)} = \frac{f(w)}{\bar{F}(w)} + \frac{g(w)}{\bar{G}(w)}.$$

So then, this would yield

$$h_U(w) = \frac{e^{-\eta w}\lambda e^{-\lambda w} + e^{-\lambda w}\eta e^{-\eta w}}{e^{-w(\lambda+\eta)}} = \frac{\lambda e^{-\lambda w}}{e^{-\lambda w}} + \frac{\eta e^{-\eta w}}{e^{-\eta w}} = \lambda + \eta.$$

*1.3.5*

For the random variable $\max\{X,\ Y\}$, the CDF $K$ is

$$K(w) = F(w)G(w).$$

Therefore, we have that

$$K(w) = (1 - e^{-\lambda w})(1 - e^{-\eta w}) = 1 - e^{-\eta w} - e^{-\lambda w} + e^{-(\lambda+\eta)w}.$$

*1.3.6*

Then, we have that the density of $\max\{X,\ Y\}$ can be found using $K$. So we have that

$$\begin{aligned}k(w) &= \frac{d}{dw}K(w) = \eta e^{-\eta w} + \lambda e^{-\lambda w} - (\lambda+\eta)e^{-(\lambda+\eta)w} \\ &= \eta e^{-\eta w} + \lambda e^{-\lambda w} - (\lambda+\eta)e^{-(\lambda+\eta)w} \\ &= \lambda e^{-\lambda w}(1 - e^{-\eta w}) + \eta e^{-\eta w}(1 - e^{-\lambda w}) \\ &= f(w)G(w) + g(w)F(w).\end{aligned}$$

*1.3.7*

For the hazard rate of $\max\{X,\ Y\}$, we have that it is

$$h_K(w) = \frac{k(w)}{\bar{K}(w)} = \frac{f(w)G(w) + g(w)F(w)}{1 - F(w)G(w)}.$$

Therefore, we have that

$$h_K(w) = \frac{(1 - e^{-\lambda w})(1 - e^{-\eta w})}{1 - (\lambda e^{-\lambda w}(1 - e^{-\eta w}) + \eta e^{-\eta w}(1 - e^{-\lambda w}))}.$$

*1.3.8*

The reverse hazard function $h_{rev}(w)$ of $\max\{X,\ Y\}$ can be written as

$$h_{rev}(w) = \frac{k(w)}{K(w)} = \frac{f(w)}{F(w)} + \frac{g(w)}{G(w)}.$$

Thus, it can be written in terms of the hazard functions of $X$ and $Y$,

$$h_{rev}(w) = \frac{\lambda e^{-\lambda w}}{(1-e^{-\lambda w})} + \frac{\eta e^{-\eta w}}{(1-e^{-\eta w})}.$$

*1.3.9*

Let $Z = X + Y$, where $Z$ has CDF $T$. Then

$$\begin{aligned}
T(w) &= \int_0^\infty F(w-u)dG(u) \\
&= \int_0^w F(w-u)dG(u) \\
&= \int_0^w (1-e^{-\lambda(w-u)}) \left(\frac{d}{du}(1-e^{-\eta w})\right) \\
&= \int_0^w \left(1-e^{-\lambda(w-u)}\right) \eta e^{-\eta u} du \\
&= \eta \int_0^w \left(e^{-\eta u} - e^{-\lambda(w-u)} e^{-\eta u}\right) du \\
&= \eta \int_0^w e^{-\eta u} du - \eta \int_0^w e^{-\lambda w} e^{(\lambda-\eta)u} du \\
&= \eta \left[\frac{-1}{\eta} e^{-\eta u}\right]_0^w - \eta e^{-\lambda w} \int_0^w e^{(\lambda-\eta)u} du \\
&= 1 - e^{-\eta w} - \eta e^{-\lambda w} \left[\frac{1}{\lambda-\eta}\left(e^{(\lambda-\eta)u}\right)\right]_0^w \\
&= 1 - e^{-\eta w} - \frac{\eta}{\lambda-\eta}\left(1-e^{-(\lambda-\eta)w}\right), \quad \lambda \neq \eta.
\end{aligned}$$

*1.3.10*

Given the CDF's of $F$, $G$, and $K$ are continuous,

$$t(w) = \int_0^w f(w-u)g(u)du.$$

So then we have that

$$t(w) = \int_0^w \lambda e^{-\lambda(w-u)} \eta e^{-\eta u} du.$$

We can simplify the integral step by step:

$$\begin{aligned}
t(w) &= \lambda\eta \int_0^w e^{-\lambda(w-u)} e^{-\eta u} du \\
&= \lambda\eta \int_0^w e^{-\lambda w} e^{\lambda u} e^{-\eta u} du \\
&= \lambda\eta e^{-\lambda w} \int_0^w e^{(\lambda-\eta)u} du \\
&= \lambda\eta e^{-\lambda w} \left[ \frac{1}{\lambda-\eta} e^{(\lambda-\eta)u} \right]_0^w, \quad \lambda \neq \eta \\
&= \lambda\eta e^{-\lambda w} \left( \frac{1}{\lambda-\eta} \left( e^{(\lambda-\eta)w} - 1 \right) \right) \\
&= \frac{\lambda\eta}{\lambda-\eta} \left( e^{-\eta w} - e^{-\lambda w} \right), \quad \lambda \neq \eta.
\end{aligned}$$