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Course: STA 5385
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Homework 2

1.

Let

$$\mathbf{X} = (X_1, X_2)' \sim \mathcal{N}_2\left(\boldsymbol{\mu} = (\mu_1, \mu_2)', \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\right).$$

1a.

We have that $\mathbf{Y} = \mathbf{AX} = (X_1 - X_2, X_1 + X_2)'$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

By properties of bivariate normal vectors, if A is a $q \times p$ constant matrix then $\mathbf{AX} \sim \mathcal{N}_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A\Sigma A}')$. So then $\mathbf{Y} \sim \mathcal{N}_2(\mathbf{A}\boldsymbol{\mu}, \mathbf{A\Sigma A}')$, where

$$\mathbf{A}\boldsymbol{\mu} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 \end{bmatrix},$$

$$\mathbf{A\Sigma A}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \sigma_{11} + 2\sigma_{12} + \sigma_{22} & \sigma_{11} - \sigma_{22} \\ \sigma_{11} - \sigma_{22} & \sigma_{11} - 2\sigma_{12} + \sigma_{22} \end{bmatrix}.$$

1b.

Since $X_1 - X_2$ and $X_1 + X_2$ are univariate normal, as each is a linear combination of univariate normal random variables, $\text{Cov}(X_1 - X_2, X_1 + X_2) = 0 \iff X_1 - X_2$ and $X_1 + X_2$ are independent. If $\text{Var}(X_1) = \text{Var}(X_2)$, then $\sigma_{11} = \sigma_{22}$. So then the covariance matrix of $\mathbf{Y}$ would be

$$\begin{bmatrix} \sigma_{11} + 2\sigma_{12} + \sigma_{22} & \sigma_{11} - \sigma_{22} \\ \sigma_{11} - \sigma_{22} & \sigma_{11} - 2\sigma_{12} + \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{11} + 2\sigma_{12} + \sigma_{22} & 0 \\ 0 & \sigma_{11} - 2\sigma_{12} + \sigma_{22} \end{bmatrix}.$$

Here we have the $\text{Var}(X_1) = \text{Var}(X_2) \iff \text{Cov}(X_1 - X_2, X_1 + X_2) = 0 \iff X_1 - X_2$ and $X_1 + X_2$ are independent.

1c.

Supposing $\text{Var}(X_1) \neq \text{Var}(X_2)$, we can find a constant a such that $X_1 - aX_2$ and $X_1 + aX_2$ are independent. First, we solve the following system for a

$$\begin{bmatrix} 1 & -a \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -a & 1 \end{bmatrix} = \begin{bmatrix} \text{Var}(X_1 - aX_2) & 0 \\ 0 & \text{Var}(X_1 + aX_2) \end{bmatrix}$$

$$\implies \begin{bmatrix} \sigma_{11} - 2a\sigma_{12} + a^2\sigma_{22} & \sigma_{11} + \sigma_{12} - a(\sigma_{12} + \sigma_{22}) \\ \sigma_{11} + \sigma_{12} - a(\sigma_{12} + \sigma_{22}) & \sigma_{11} + 2\sigma_{12} + \sigma_{22} \end{bmatrix} = \begin{bmatrix} \text{Var}(X_1 - aX_2) & 0 \\ 0 & \text{Var}(X_1 + aX_2) \end{bmatrix}$$

$$\implies 0 = \sigma_{11} + \sigma_{12} - a(\sigma_{12} + \sigma_{22})$$

$$\implies a = \frac{\sigma_{11} + \sigma_{12}}{\sigma_{22} + \sigma_{12}}.$$

Hence, the constant a that makes $X_1 - aX_2$ and $X_1 + aX_2$ independent is

$$a = \frac{\text{Var}(X_1) + \text{Cov}(X_1, X_2)}{\text{Var}(X_2) + \text{Cov}(X_1, X_2)}.$$

2.

Suppose $X_1 \sim \mathcal{N}(\mu = 5, \sigma^2 = 2)$, and let

$$X_2 = 2X_1 + \epsilon_1$$
$$X_3 = 3 - X_1 + \epsilon_2$$

Where ϵ_1 and ϵ_2 are independent and distributed $\mathcal{N}(0, 1)$, and both are also independent of X_1.

2a.

We have that the random vector $\mathbf{Y} = (X_1, X_2, X_3)'$ can be written as

$$\mathbf{Y} = \mathbf{AW} + \mathbf{d}.$$

Where $\mathbf{A}$ is a 3×3 constant matrix, $\mathbf{W}$ is a 3-dimensional random vector, and $\mathbf{d}$ is a three dimensional constant vector, all written as

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} X_1 \\ \epsilon_1 \\ \epsilon_2 \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

We have that $\mathbf{W}$ is a random vector with independent normally distributed entries, implying that

$$\mathbf{W} \sim \mathcal{N}_3 \left(\boldsymbol{\mu}_{\mathbf{W}} = (5, 0, 0)', \boldsymbol{\Sigma}_{\mathbf{W}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

Then, if $\mathbf{A}$ is a $q \times p$ constant matrix then $\mathbf{AW} \sim \mathcal{N}_3(\mathbf{A}\boldsymbol{\mu}_{\mathbf{W}}, \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{W}}\mathbf{A}')$, so

$$\mathbf{AW} \sim \mathcal{N}_3 \left(\boldsymbol{\mu}_{\mathbf{AW}} = (5, 10, -2)', \boldsymbol{\Sigma}_{\mathbf{W}} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -4 \\ -2 & -4 & 3 \end{bmatrix} \right).$$

Then, we have it that if $\mathbf{d}$ is a p-dimensional constant vector, then $\mathbf{AW} + \mathbf{d} \sim \mathcal{N}_p(\boldsymbol{\mu} + \mathbf{d}, \boldsymbol{\Sigma})$. So then since $\mathbf{Y} = \mathbf{AW} + \mathbf{d}$,

$$\mathbf{Y} \sim \mathcal{N}_3 \left(\boldsymbol{\mu}_{\mathbf{Y}} = (5, 10, 1)', \boldsymbol{\Sigma}_{\mathbf{Y}} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -4 \\ -2 & -4 & 3 \end{bmatrix} \right).$$

2b.

For X_2 and X_3, we have that $\text{Cor}(X_2, X_3) = \dfrac{\sigma_{23}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{33}}} = \dfrac{-4}{3\sqrt{3}} = -0.77$. This can be considered to be a strong negative linear relationship in some contexts, though a traditional threshold for classifying a 'strong' linear relationship is usually $|r| \geq 0.85$.

2c.

In finding the conditional distribution of $(X_2, X_3)'$ given X_1, we partition $\mathbf{Y}$ into $\mathbf{Y}_1 = (X_1)$ and $\mathbf{Y}_2 = (X_2, X_3)'$, $\boldsymbol{\mu}_{\mathbf{Y}}$ into $\boldsymbol{\mu}_1 = (5)$ and $\boldsymbol{\mu}_2 = (10, 1)'$, and lastly $\boldsymbol{\Sigma}_{\mathbf{Y}}$ into $\boldsymbol{\Sigma}_{11} = 2, \boldsymbol{\Sigma}_{12} = (4, -2), \boldsymbol{\Sigma}_{21} = (4, -2)'$, and $\boldsymbol{\Sigma}_{22}$ as a 2×2 matrix with column vectors $(9, -4)'$ and $(-4, 3)'$.

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} \qquad \boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \qquad \boldsymbol{\Sigma}_{\mathbf{Y}} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

Since $|\boldsymbol{\Sigma}_{22}| = 11$, we have that the conditional distribution of $\mathbf{Y}_2$ given $\mathbf{Y}_1 = \mathbf{y}_1$ is bivariate normal with mean vector

$$\begin{aligned} \boldsymbol{\mu}_{2|1} &= \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1) \\ &= \begin{bmatrix} 10 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \end{bmatrix} (\mathbf{y}_1 - 5) \\ &= \begin{bmatrix} 10 \\ 1 \end{bmatrix} + \begin{bmatrix} 2\mathbf{y}_1 - 10 \\ 5 - \mathbf{y}_1 \end{bmatrix} \\ &= \begin{bmatrix} 2\mathbf{y}_1 \\ 6 - \mathbf{y}_1 \end{bmatrix}, \end{aligned}$$

and covariance matrix

$$\begin{aligned} \boldsymbol{\Sigma}_{2|1} &= \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} \\ &= \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \mathbf{I}_2. \end{aligned}$$

Hence, the conditional distribution of $\mathbf{Y}_2 | \mathbf{Y}_1 \sim \mathcal{N}_2\left(\boldsymbol{\mu}_{2|1} = (2\mathbf{y}_1, 6 - \mathbf{y}_1)', \boldsymbol{\Sigma}_{2|1} = \mathbf{I}_2\right)$. X_2 and X_3 are not conditionally independent of X_1 since $\boldsymbol{\Sigma}_{21} \neq \mathbf{0}$ and $\boldsymbol{\Sigma}_{12} \neq \mathbf{0}$.

3.

Let $X_1 \sim \mathcal{N}(0, 1)$ and then define

$$X_2 = \begin{cases} -X_1 & -1 \leq X_1 \leq 1 \\ X_1 & \text{otherwise.} \end{cases}$$

3a.

We will show that $X_2 \sim \mathcal{N}(0, 1)$. By the jacobian method, we have that when $-1 \leq X_1 \leq 1$, $X_2 = g(X_1) = -X_1$ the PDF of X_2

$$f_{X_2}(x_2) = f_{X_1}(X_2)\left|\frac{\partial}{\partial X_2} g^{-1}(X_2)\right| = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}(-x_2)^2} |-1| = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}(x_2)^2} = f_{X_1}(x_2).$$

So when $-1 \leq X_1 \leq 1$, $X_2 \sim \mathcal{N}(0, 1)$ since $f_{X_2}(x) = f_{X_1}(x)$. Also, when $X_1 \notin [-1, 1]$, the PDFs of X_1 and X_2 match. $\therefore X_2 \sim N(0, 1)$.

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3b.

If $(X_1, X_2)'$ was bivariate normal, then any linear combination of its elements would be univariate normal. Consider the linear combination

$$X_1 + X_2 = \begin{cases} X_1 - X_1 = 0 & -1 \leq X_1 \leq 1 \\ X_1 + X_1 = 2X_1 & \text{otherwise.} \end{cases}$$

This implies that $P(-1 \leq X_1 + X_2 \leq 1) = P(X_1 + X_2 = 0)$. So then we can say that $X_1 + X_2$ is a mixture of a distribution at 0, and a $\mathcal{N}(0, 4)$ distribution, which is not a normal distribution. Therefore, we have that since at least one linear combination of the elements of $(X_1, X_2)'$ is not normally distributed, we cannot say that $(X_1, X_2)'$ is jointly normally distributed.

4.

Specifying the following distributions, we have:

a) Since $(\mathbf{X}_1 - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu})'$ is a quadratic form of a multivariate normal distribution, we know that $(\mathbf{X}_1 - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu})' \sim \chi^2_5$.

b) From the notes we have that $\bar{\mathbf{X}} \sim \mathcal{N}_5\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right)$.

c) By the Multivariate Central Limit Theorem, $\sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim N_5(0, \boldsymbol{\Sigma})$.

d) The distribution of $(n-1)\mathbf{S}$ is $W(n-1, \boldsymbol{\Sigma})$, so in this particular problem, $(n-1)\mathbf{S} \sim W(14, \boldsymbol{\Sigma})$, as $n = 15$.

e) Since $\mathbf{B}$ is a 2×5 matrix with more columns than rows, and $14\mathbf{S} \sim W(14, \boldsymbol{\Sigma})$, then $\mathbf{B}(14\boldsymbol{\Sigma})\mathbf{B}' \sim W(14, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}')$.