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Homework 1

1a.

Let $\mathbf{A}$ be a symmetric $n \times n$ be a positive definite matrix. By the eigenvalue decomposition, we have that $\mathbf{A}$ can be decomposed into three matrices,

$$\mathbf{A} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}',$$

where $\mathbf{E}$ is an orthogonal matrix with the columns as the normalized eigenvectors of $\mathbf{A}$, and $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues of $\mathbf{A}$ in descending order. We have then that $\mathbf{A} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}'$ satisfies

$$\mathbf{x}'\mathbf{E}\mathbf{\Lambda}\mathbf{E}'\mathbf{x} > 0,$$

by definition of positive definite, where $\mathbf{x}$ is any vector except the zero vector. Let $\mathbf{y} = \mathbf{E}'\mathbf{x}$. Then we have that

$$\mathbf{x}'\mathbf{E}\mathbf{\Lambda}\mathbf{E}'\mathbf{x} = \mathbf{y}'\mathbf{\Lambda}\mathbf{y} = \sum_{i=1}^{n} \lambda_i y_i^2 > 0. \tag{1}$$

Since $y_i^2 > 0$, then all $\lambda_i > 0$ for $i = 1, \ldots, n$. So if $\mathbf{A}$ is a symmetric $n \times n$ be a positive definite matrix, that implies all its eigenvalues are positive.

Starting from the proposition that $\mathbf{A}$ is a symmetric $n \times n$ matrix such that all of its eigenvalues are positive, we can still apply the same method with eigenvalue decomposition to arrive at (1), where it is already known that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive, and it is trivial that $y_i^2 > 0$. Therefore, $\mathbf{A}$ satisfies the conditions of a positive definite matrix.

Hence symmetric $n \times n$ matrix $\mathbf{A}$ is positive definite $\iff$ all its eigenvalues are positive.

1b.

Given matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

We find the eigenvalues by $\det(\mathbf{A} - \lambda \mathbf{I}_3) = 0$

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}_3) &= \det\left(\begin{bmatrix} 8-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{bmatrix}\right) \\
&= (8-\lambda)\det\begin{bmatrix} 2-\lambda & 0 \\ 0 & 8-\lambda \end{bmatrix} + (4)\det\begin{bmatrix} 0 & 2-\lambda \\ 4 & 0 \end{bmatrix} \\
&= (8-\lambda)^2(2-\lambda) - 4(4(2-\lambda)) \\
&= -\lambda^3 + 18\lambda^2 - 80\lambda + 96 \\
&= (\lambda - 12)(\lambda - 4)(\lambda - 2).
\end{aligned}$$

Here we see the eigenvalues of $\mathbf{A}$ are 12, 4, and 2, all of which are positive. So $\mathbf{A}$ is positive definite.

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1c.

Finding the spectral decomposition, of $\mathbf{A}$, we have that the eigenvalues are 12, 4, and 2 respectively. To find the corresponding eigenvector for $\lambda = 12$, we want to solve the system

$$\begin{bmatrix} 8-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{bmatrix} = \begin{bmatrix} -4 & 0 & 4 \\ 0 & -10 & 0 \\ 4 & 0 & -4 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Letting $e_{11} = 1$, we have the system:

$$\begin{aligned} -4 + 4e_{31} &= 0 \\ -10e_{21} &= 0 \\ 4 - 4e_{31} &= 0 \end{aligned}$$

yielding the eigenvector for $\lambda = 12$, $\mathbf{e}_1 = (1, 0, 1)'$. To find the corresponding eigenvector for $\lambda = 4$, we want to solve the system

$$\begin{bmatrix} 8-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{bmatrix} \begin{bmatrix} e_{12} \\ e_{22} \\ e_{32} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 4 \\ 0 & -2 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} e_{12} \\ e_{22} \\ e_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Letting $e_{12} = 1$, we have the system:

$$\begin{aligned} 4 + 4e_{32} &= 0 \\ -2e_{22} &= 0 \\ 4 + 4e_{32} &= 0 \end{aligned}$$

yielding the eigenvector for $\lambda = 4$, $\mathbf{e}_2 = (1, 0, -1)'$. Finally, to find the corresponding eigenvector for $\lambda = 2$, we want to solve the system

$$\begin{bmatrix} 8-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 4 & 0 & 8-\lambda \end{bmatrix} \begin{bmatrix} e_{13} \\ e_{23} \\ e_{33} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} e_{13} \\ e_{23} \\ e_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We then have the system:

$$\begin{aligned} 6e_{13} + 4e_{33} &= 0 \\ 4e_{13} + 6e_{33} &= 0 \end{aligned}$$

which is only satisfied when $e_{13} = e_{33} = 0$, with e_{23} as a free parameter, so we let it equal 1. This yields the eigenvector for $\lambda = 2$, $\mathbf{e}_3 = (0, 1, 0)'$. We normalize $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ and set them as the columns of the matrix $\mathbf{E}$, yielding the spectral decomposition of $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}' = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}.$$

1d.

Since $\mathbf{A}$ is positive definite, we can take the square root of the matrix $\mathbf{\Lambda}$, which to find the square root of $\mathbf{A}$. So then we have $\mathbf{A} = \mathbf{E}\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{E}'$, which yields

$$\mathbf{A}^{\frac{1}{2}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{3} & 0 & -1+\sqrt{3} \\ 0 & \sqrt{2} & 0 \\ -1+\sqrt{3} & 0 & 1+\sqrt{3} \end{bmatrix}.$$

2a.

Let $\mathbf{Q}$ be a $p \times p$ orthogonal matrix. By definition of an orthogonal matrix, and we have that $\mathbf{Q}\mathbf{Q}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_p$. Likewise, we have that

$$\det(\mathbf{Q})\det(\mathbf{Q}') = \det(\mathbf{Q}')\det(\mathbf{Q}) = (\det(\mathbf{Q}))^2 = \det(\mathbf{I}_p) = 1.$$

Since $(\det(\mathbf{Q}))^2 = 1$, $\det(\mathbf{Q}) = \pm 1$.

2b.

Let $\mathbf{A}$ also be a $p \times p$ matrix, though $\mathbf{A}$ is not orthogonal as $\mathbf{Q}$ is. We have that since $\mathbf{Q}$ is orthogonal, then $\det(\mathbf{Q}) = \det(\mathbf{Q}') = \pm 1$. Consider the determinant of $\mathbf{A}$, the product of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$.

$$\begin{aligned} \det(\mathbf{A}) = \det(\mathbf{E}\mathbf{\Lambda}\mathbf{E}') = \det(\mathbf{E})\det(\mathbf{\Lambda})\det(\mathbf{E}') &= 1 \cdot \det(\mathbf{\Lambda}) \cdot 1 \\ &= \prod_{i=1}^{n} \lambda_i \\ &= 1 \cdot 1 \cdot \det(\mathbf{\Lambda}) \cdot 1 \cdot 1 = \det(\mathbf{Q}')\det(\mathbf{E})\det(\mathbf{\Lambda})\det(\mathbf{E}')\det(\mathbf{Q}) \\ &= \det(\mathbf{Q}'\mathbf{E}\mathbf{\Lambda}\mathbf{E}'\mathbf{Q}) = \det(\mathbf{Q}'\mathbf{A}\mathbf{Q}). \end{aligned}$$

In both cases, since $\det(\mathbf{E}) = \det(\mathbf{E}')$ and $\det(\mathbf{Q}) = \det(\mathbf{Q}')$, all negatives cancel. Therefore, since for any $p \times p$ matrix, the determinant is the product of its eigenvalues, and since

$$\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i = \det(\mathbf{Q}'\mathbf{A}\mathbf{Q}),$$

then $\mathbf{A}$ and $\mathbf{Q}'\mathbf{A}\mathbf{Q}$ have the same eigenvalues.

3.

Consider a random vector $\mathbf{X} = (X_1, X_2, X_3)'$ with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ and covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

We know that the mean of $g(\mathbf{X})$ is:

$$\mathbb{E}[g(\mathbf{X})] = \mathbb{E}[\mathbf{c}'\mathbf{X}] = \mathbf{c}^{\mathbb{E}}[\mathbf{X}] = \mathbf{c}'\boldsymbol{\mu} \tag{2}$$

where $\boldsymbol{\mu} = \mathbb{E}[\mathbf{X}]$ is the mean vector of $\mathbf{X}$. Also, the variance of $g(\mathbf{X})$ is:

$$\mathrm{Var}(g(\mathbf{X})) = \mathrm{Var}(\mathbf{c}'\mathbf{X}) = \mathbf{c}'\mathbf{\Sigma}\mathbf{c} \tag{3}$$

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a.

The random variable $X_1 + 3X_2$ can be characterized as $\mathbf{c}'\mathbf{X}$, where $\mathbf{c} = (1, 3, 0)$. Therefore, by (2), we have that $\mathbb{E}[\mathbf{c}'\mathbf{X}] = \mathbf{c}'\mathbb{E}[\mathbf{X}] = \mathbf{c}'\boldsymbol{\mu} = \mu_1 + 3\mu_2$. By (3), the variance of $X_1 + 3X_2$ is $\mathbf{c}'\boldsymbol{\Sigma}\mathbf{c}$, which can be expressed as

$$\mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \sigma_{11} + 3\sigma_{21} + 3(\sigma_{12} + 3\sigma_{22}).$$

Since $\boldsymbol{\Sigma}$ is symmetric, we have that the variance simplifies to $\sigma_{11} + 6\sigma_{12} + 9\sigma_{22}$.

b.

The random variable $-X_1 + 2X_2$ can be characterized as $\mathbf{c}'\mathbf{X}$, where $\mathbf{c} = (-1, 2, 0)$. Therefore, by (2), we have that $\mathbb{E}[\mathbf{c}'\mathbf{X}] = -\mu_1 + 2\mu_2$. By (3), the variance of $-X_1 + 2X_2$ can be expressed as

$$\mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \sigma_{11} - 2(\sigma_{21} + \sigma_{12}) + 4\sigma_{22}.$$

Since $\boldsymbol{\Sigma}$ is symmetric, we have that the variance simplifies to $\sigma_{11} - 4\sigma_{12} + 4\sigma_{22}$.

c.

The random variable $X_1 + X_2 - X_3$ can be characterized as $\mathbf{c}'\mathbf{X}$, where $\mathbf{c} = (1, 1, -1)$. Therefore, by (2), we have that $\mathbb{E}[\mathbf{c}'\mathbf{X}] = \mu_1 + \mu_2 - \mu_3$. Also by (3), the variance of $X_1 + X_2 - X_3$ can be expressed as

$$\begin{aligned} \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} &= \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \sigma_{11} + \sigma_{21} - \sigma_{31} + \sigma_{12} + \sigma_{22} - \sigma_{32} - \sigma_{13} - \sigma_{23} + \sigma_{33}. \end{aligned}$$

Since $\boldsymbol{\Sigma}$ is symmetric, we have that the variance simplifies to $\sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23}$.

d.

The random variable $X_1 + 2X_2 + X_3$ can be characterized as $\mathbf{c}'\mathbf{X}$, where $\mathbf{c} = (1, 2, 1)$. Therefore, by (2), we have that $\mathbb{E}[\mathbf{c}'\mathbf{X}] = \mu_1 + 2\mu_2 + \mu_3$. Also by (3), the variance of $X_1 + 2X_2 + X_3$ can be expressed as

$$\begin{aligned} \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} &= \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \sigma_{11} + 2\sigma_{21} + \sigma_{31} + 2(\sigma_{12} + 2\sigma_{22} + \sigma_{23}) + \sigma_{13} + 2\sigma_{23} + \sigma_{33}. \end{aligned}$$

Since $\boldsymbol{\Sigma}$ is symmetric, we have that the variance simplifies to $\sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} + 2\sigma_{13} + 4\sigma_{23}$.

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e.

The random variable $3X_2 + 2X_3$ can be characterized as $\mathbf{c}'\mathbf{X}$, where $\mathbf{c} = (0, 3, 2)$. Therefore, by (2), we have that $\mathbb{E}[\mathbf{c}'\mathbf{X}] = 3\mu_2 + 2\mu_3$. Also by (3), the variance of $3X_2 + 2X_3$ can be expressed as

$$\mathbf{c}'\mathbf{\Sigma}\mathbf{c} = \begin{bmatrix} 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = 3(3\sigma_{22} + 2\sigma_{23})) + 2(3\sigma_{32} + 2\sigma_{33}).$$

Since X_2 and X_3 are independent, we have that σ_{23} and σ_{32} are zero, which mean the variance simplifies to $9\sigma_{22} + 4\sigma_{33}$.

4a.

Considering the matrix,

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 4 & 12 \\ 4 & 2 & 4 \\ 7 & 0 & 6 \\ 5 & 3 & 8 \end{bmatrix},$$

The deviation vectors for $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ are

$$\mathbf{d}_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{d}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad \mathbf{d}_1 = \begin{bmatrix} -6 \\ 6 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

To show their linear dependence, we form matrix $\mathbf{D}$ with $\mathbf{d}_1, \mathbf{d}_2$, and $\mathbf{d}_3$ as the column vectors and perform gaussian elimination, yielding

$$\begin{bmatrix} 2 & 1 & -6 & | & 0 \\ 0 & 3/2 & 9 & | & 0 \\ 0 & 0 & -8 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This system has infinitely many solutions, implying the column vectors are linearly dependent.

4b.

We can find the sample covariance $\mathbf{S} = \frac{1}{n-1}\mathbf{D}'\mathbf{D}$,

$$\mathbf{S} = \frac{1}{4}\mathbf{D}'\mathbf{D} = \frac{1}{4} \begin{bmatrix} -2 & 1 & -1 & 2 & 0 \\ 1 & 2 & 0 & -2 & 1 \\ -6 & 6 & -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & -6 \\ 1 & 2 & 6 \\ -1 & 0 & -2 \\ 2 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2.5 & -1 & 5 \\ -1 & 2.5 & 2 \\ 5 & 2 & 20 \end{bmatrix}$$

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4c.

The total sample variance is $\text{tr}(\mathbf{S}) = 2.5 + 2.5 + 20 = 25$.