STA 6384, Report 2.4

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Problem: Work problem 2.3, p. 61 of Agresti.

An article in *The New York Times* (Feb. 17, 1999) about the PSA blood test for detecting prostate cancer stated: "The test fails to detect prostate cancer in 1 in 4 men who have the disease (false-negative results), and as many as two-thirds of the men tested receive false-positive results.'' Let $C(\bar{C})$ denote the event of having (not having) prostate cancer, and let $+(-)$ denote a positive (negative) test result. Which is true: $P(-|C) = \frac{1}{4}$ or $P(C|-) = \frac{1}{4}$? $P(\bar{C}|+) = \frac{2}{3}$ or $P(+|\bar{C}) = \frac{2}{3}$? Determine the sensitivity and specificity.

Let's define the events based on the problem description:

- C: The event that a man has prostate cancer.
- $\bar{C}$: The event that a man does not have prostate cancer.
- $+$: The event of a positive test result.
- $-$: The event of a negative test result.

The article from *The New York Times* provides two key statistics:

1. **False-Negative Rate**: "The test fails to detect prostate cancer in 1 in 4 men who have the disease." A false negative is a negative test result ($-$) for a person who has the disease (C). This corresponds to the conditional probability $P(-|C)$.

$$P(-|C) = \frac{1}{4}$$

2. **False-Positive Rate**: "...as many as two-thirds of the men tested receive false-positive results." A false positive is a positive test result ($+$) for a person who does not have the disease ($\bar{C}$). This corresponds to the conditional probability $P(+|\bar{C})$.

$$P(+|\bar{C}) = \frac{2}{3}$$

Answering the Questions

Which statement about the false-negative rate is true?

The question asks to choose between $P(-|C) = \frac{1}{4}$ and $P(C|-) = \frac{1}{4}$. As established from the text, the false-negative rate is the probability of getting a negative result given the person has cancer. Therefore, the correct statement is:

$$\mathbf{P}(-|\mathbf{C}) = \frac{\mathbf{1}}{\mathbf{4}}$$

The alternative, $P(C|-)$, is the probability of having cancer given a negative test result, which is known as the post-test probability of disease after a negative result.

Which statement about the false-positive rate is true?

The question asks to choose between $P(C|\bar{+}) = \frac{2}{3}$ and $P(+|\bar{C}) = \frac{2}{3}$. The text defines the false-positive rate as the fraction of men without cancer who receive a positive test. This is exactly the definition of $P(+|\bar{C})$. Therefore, the correct statement is:

$$\mathbf{P}(+|\bar{\mathbf{C}}) = \frac{\mathbf{2}}{\mathbf{3}}$$

The notation $P(C|\bar{+})$ is non-standard, but if interpreted as $P(\bar{C}|+)$, it would represent the post-test probability of not having cancer after a positive result.

Determining Sensitivity and Specificity

Sensitivity

Sensitivity, or the true positive rate, is the probability that the test correctly identifies a person who has the disease. It is the probability of a positive test ($+$) given that the person has cancer (C).

$$\text{Sensitivity} = P(+|C)$$

For any person with cancer, the test result must be either positive or negative, so the probabilities must sum to 1:

$$P(+|C) + P(-|C) = 1$$

Using the known false-negative rate, we can solve for the sensitivity:

$$P(+|C) = 1 - P(-|C) = 1 - \frac{1}{4} = \frac{3}{4}$$

The sensitivity of the test is $\frac{\mathbf{3}}{\mathbf{4}}$, or $\mathbf{75}\%$.

Specificity

Specificity, or the true negative rate, is the probability that the test correctly identifies a person who does not have the disease. It is the probability of a negative test ($-$) given that the person does not have cancer ($\bar{C}$).

$$\text{Specificity} = P(-|\bar{C})$$

Similarly, for a person without cancer, the test result must be either positive or negative:

$$P(+|\bar{C}) + P(-|\bar{C}) = 1$$

Using the known false-positive rate, we can solve for the specificity:

$$P(-|\bar{C}) = 1 - P(+|\bar{C}) = 1 - \frac{2}{3} = \frac{1}{3}$$

The specificity of the test is $\mathbf{\frac{1}{3}}$, or approximately $\mathbf{33.3\%}$.