STA 6384, Report 1.9

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Problem: Work problem 1.11, p. 30 of Agresti.

A binomial experiment tests $H_0 : \pi = 0.50$ against $H_a : \pi \neq 0.50$ using significance level 0.05. Only $n = 5$ observations are available. Show that the true null probability of rejecting H_0 is 0.00 for an exact binomial test and $\frac{1}{16}$ using the large-sample score test.

Let X be the number of successes in $n = 5$ trials. Under the null hypothesis $H_0 : \pi = 0.5$, the random variable X follows a binomial distribution, $X \sim \text{Binomial}(5, 0.5)$.

The probability mass function (PMF) under H_0 is:

$$P(X = k | H_0) = \binom{5}{k}(0.5)^k(0.5)^{5-k} = \binom{5}{k}\left(\frac{1}{2}\right)^5 = \frac{1}{32}\binom{5}{k}$$

The possible outcomes for X are $\{0, 1, 2, 3, 4, 5\}$. We can calculate the probability for each outcome:

$$\begin{aligned}
P(X = 0) &= \frac{1}{32}\binom{5}{0} = \frac{1}{32} \\
P(X = 1) &= \frac{1}{32}\binom{5}{1} = \frac{5}{32} \\
P(X = 2) &= \frac{1}{32}\binom{5}{2} = \frac{10}{32} \\
P(X = 3) &= \frac{1}{32}\binom{5}{3} = \frac{10}{32} \\
P(X = 4) &= \frac{1}{32}\binom{5}{4} = \frac{5}{32} \\
P(X = 5) &= \frac{1}{32}\binom{5}{5} = \frac{1}{32}
\end{aligned}$$

For a two-sided exact binomial test, the p-value is the sum of probabilities of outcomes as extreme or more extreme than the one observed. The test is symmetric around the expected value $E[X] = n\pi_0 = 5(0.5) = 2.5$. We reject H_0 if the p-value is less than or equal to $\alpha = 0.05$.

Let's find the p-value for the most extreme possible outcomes, $X = 0$ or $X = 5$. Due to symmetry, they will have the same p-value.

$$\begin{aligned}
p\text{-value for } X = 0 &= P(X \leq 0) + P(X \geq 5) \\
&= P(X = 0) + P(X = 5) \\
&= \frac{1}{32} + \frac{1}{32} = \frac{2}{32} = 0.0625
\end{aligned}$$

Since the p-value for the most extreme outcomes (0.0625) is greater than the significance level $\alpha = 0.05$, we can never reject the null hypothesis, regardless of the outcome. Any less extreme outcome (e.g., $X = 1$ or $X = 4$) will have an even larger p-value.

The rejection region is the set of outcomes for which we would reject H_0. In this case, the rejection region is the empty set, $\emptyset$.

Therefore, the true null probability of rejecting H_0 is the probability of an outcome falling in the rejection region, which is $P(X \in \emptyset) = \mathbf{0.00}$.

The large-sample score test statistic for a proportion is:

$$z = \frac{\hat{\pi} - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}$$

where $\hat{\pi} = X/n$, $\pi_0 = 0.5$, and $n = 5$.

$$z = \frac{X/5 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{5}}} = \frac{X/5 - 0.5}{\sqrt{\frac{0.25}{5}}} = \frac{X/5 - 0.5}{\sqrt{0.05}} \approx \frac{X/5 - 0.5}{0.2236}$$

For a two-sided test with $\alpha = 0.05$, the rejection rule is to reject H_0 if $|z| \geq z_{\alpha/2} = z_{0.025} = 1.96$.

We check this condition for each possible value of X:

- For $X = 0$: $z = \frac{0-0.5}{\sqrt{0.05}} \approx -2.236$. Since $|-2.236| > 1.96$, we **reject** H_0.
- For $X = 1$: $z = \frac{1/5-0.5}{\sqrt{0.05}} = \frac{-0.3}{\sqrt{0.05}} \approx -1.342$. Since $|-1.342| < 1.96$, we do not reject H_0.
- For $X = 2$: $z = \frac{2/5-0.5}{\sqrt{0.05}} = \frac{-0.1}{\sqrt{0.05}} \approx -0.447$. Since $|-0.447| < 1.96$, we do not reject H_0.
- By symmetry, for $X = 3$, $z \approx 0.447$ (do not reject). For $X = 4$, $z \approx 1.342$ (do not reject).
- For $X = 5$: $z = \frac{5/5-0.5}{\sqrt{0.05}} = \frac{0.5}{\sqrt{0.05}} \approx 2.236$. Since $|2.236| > 1.96$, we **reject** H_0.

The rejection region for the score test consists of the outcomes $\{0, 5\}$.

The true null probability of rejecting H_0 is the probability of observing an outcome in this rejection region, calculated using the true binomial distribution under H_0:

$$P(\text{Reject } H_0) = P(X \in \{0, 5\}) = P(X = 0) + P(X = 5)$$

Using the probabilities calculated earlier:

$$P(\text{Reject } H_0) = \frac{1}{32} + \frac{1}{32} = \frac{2}{32} = \mathbf{\frac{1}{16}}$$

This completes the proof.