

# STA 6384, Report 1.11

Carson Slater *Baylor University*

**Problem: Work problem 1.26, p. 33 of Agresti.**

A binomial sample of size  $n$  has  $y = 0$  successes.

(a)

**Show that the confidence interval for  $\pi$  based on the likelihood function is  $\left[0, 1 - \exp\left(-\frac{z_{\alpha/2}^2}{2n}\right)\right]$ . For  $\alpha = 0.05$ , use the expansion of an exponential function to show that this is approximately  $\left[0, \frac{1.92}{n}\right]$ .**

For a binomial sample of size  $n$  with  $y$  successes, the likelihood function for the proportion  $\pi$  is:

$$L(\pi) = \binom{n}{y} \pi^y (1 - \pi)^{n-y}$$

Given that we have  $y = 0$  successes, the likelihood function simplifies to:

$$L(\pi) = (1 - \pi)^n$$

The log-likelihood function,  $l(\pi)$ , is therefore:

$$l(\pi) = \ln(L(\pi)) = n \ln(1 - \pi)$$

For  $y = 0$ , the MLE is  $\hat{\pi} = y/n = 0$ . The log-likelihood evaluated at the MLE is  $l(\hat{\pi}) = l(0) = n \ln(1-0) = 0$ .

A  $(1 - \alpha)100\%$  confidence interval based on the likelihood ratio test is the set of all values  $\pi$  that satisfy the inequality:

$$-2[l(\pi) - l(\hat{\pi})] \leq \chi_{1,\alpha}^2$$

where  $\chi_{1,\alpha}^2$  is the upper- $\alpha$  critical value from a chi-squared distribution with 1 degree of freedom. We also know that  $\chi_{1,\alpha}^2 = (z_{\alpha/2})^2$ , where  $z_{\alpha/2}$  is the upper- $\alpha/2$  critical value from the standard normal distribution.

Substituting our functions into the inequality:

$$\begin{aligned} -2[n \ln(1 - \pi) - 0] &\leq (z_{\alpha/2})^2 \\ -2n \ln(1 - \pi) &\leq z_{\alpha/2}^2 \end{aligned}$$

Now, we solve for  $\pi$ . Dividing by  $-2n$  reverses the inequality sign:

$$\ln(1 - \pi) \geq -\frac{z_{\alpha/2}^2}{2n}$$

Exponentiating both sides preserves the inequality:

$$e^{\ln(1-\pi)} \geq \exp\left(-\frac{z_{\alpha/2}^2}{2n}\right)$$

$$1 - \pi \geq \exp\left(-\frac{z_{\alpha/2}^2}{2n}\right)$$

Finally, isolating  $\pi$  (which reverses the inequality again):

$$-\pi \geq \exp\left(-\frac{z_{\alpha/2}^2}{2n}\right) - 1$$

$$\pi \leq 1 - \exp\left(-\frac{z_{\alpha/2}^2}{2n}\right)$$

Since  $\pi$  must be non-negative, the lower bound is 0. Thus, the confidence interval is:

$$\left[0, 1 - \exp\left(-\frac{z_{\alpha/2}^2}{2n}\right)\right]$$

*Approximation for  $\alpha = 0.05$*

For  $\alpha = 0.05$ , we have  $\alpha/2 = 0.025$ . The corresponding standard normal critical value is  $z_{0.025} \approx 1.96$ .

We use the first-order Taylor series expansion for an exponential function  $e^x$  around  $x = 0$ , which is  $e^x \approx 1 + x$ . Let  $x = -\frac{z_{\alpha/2}^2}{2n}$ . For large  $n$ ,  $x$  is close to 0.

The upper bound of the confidence interval is  $1 - \exp\left(-\frac{z_{\alpha/2}^2}{2n}\right)$ . Applying the Taylor expansion:

$$1 - \exp(x) \approx 1 - (1 + x) = -x$$

Substituting  $x = -\frac{z_{\alpha/2}^2}{2n}$  back in:

$$\text{Upper Bound} \approx -\left(-\frac{z_{\alpha/2}^2}{2n}\right) = \frac{z_{\alpha/2}^2}{2n}$$

Now we plug in the value  $z_{0.025} \approx 1.96$ :

$$\text{Upper Bound} \approx \frac{(1.96)^2}{2n} = \frac{3.8416}{2n} = \frac{1.9208}{n} \approx \frac{1.92}{n}$$

Therefore, the approximate confidence interval for  $\alpha = 0.05$  is:

$$\left[0, \frac{1.92}{n}\right]$$

(b)

**For the score method, show that the confidence interval is  $\left[0, \frac{z_{\alpha/2}^2}{n+z_{\alpha/2}^2}\right]$ , or  $\left[0, \frac{3.84}{n+3.84}\right]$  when  $\alpha = 0.05$ . (See Exercise 16.30 for small-sample intervals when  $y = 0$ .)**

The score confidence interval is derived by inverting the score test. The score test statistic is:

$$Z = \frac{\hat{p} - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}}$$

where  $\hat{p} = y/n$  is the sample proportion. The set of  $\pi$  values for which  $|Z| \leq z_{\alpha/2}$  forms the  $(1 - \alpha)100\%$  confidence interval. This is equivalent to solving the inequality:

$$Z^2 = \frac{(\hat{p} - \pi)^2}{\frac{\pi(1-\pi)}{n}} \leq z_{\alpha/2}^2$$

We are given that  $y = 0$ , so the sample proportion is  $\hat{p} = 0/n = 0$ . Substituting this into the inequality gives:

$$\begin{aligned} \frac{(0 - \pi)^2}{\frac{\pi(1-\pi)}{n}} &\leq z_{\alpha/2}^2 \\ \frac{\pi^2 n}{\pi(1 - \pi)} &\leq z_{\alpha/2}^2 \end{aligned}$$

Assuming  $\pi \neq 0$  (since we are finding the interval boundary), we can simplify:

$$\frac{n\pi}{1 - \pi} \leq z_{\alpha/2}^2$$

To find the upper bound of the confidence interval, we solve the corresponding equality for  $\pi$ :

$$\begin{aligned} \frac{n\pi}{1 - \pi} &= z_{\alpha/2}^2 \\ n\pi &= z_{\alpha/2}^2(1 - \pi) \\ n\pi &= z_{\alpha/2}^2 - \pi z_{\alpha/2}^2 \end{aligned}$$

Now, we group the terms containing  $\pi$ :

$$\begin{aligned} n\pi + \pi z_{\alpha/2}^2 &= z_{\alpha/2}^2 \\ \pi(n + z_{\alpha/2}^2) &= z_{\alpha/2}^2 \\ \pi &= \frac{z_{\alpha/2}^2}{n + z_{\alpha/2}^2} \end{aligned}$$

Since the sample proportion is 0, the lower bound for the confidence interval is 0. Therefore, the score method confidence interval is:

$$\left[0, \frac{z_{\alpha/2}^2}{n + z_{\alpha/2}^2}\right]$$

*Approximation for  $\alpha = 0.05$*

For a confidence level of 95%, we have  $\alpha = 0.05$ , which gives  $\alpha/2 = 0.025$ . The critical value from the standard normal distribution is  $z_{0.025} \approx 1.96$ . The squared critical value is:

$$z_{\alpha/2}^2 \approx (1.96)^2 = 3.8416 \approx 3.84$$

Substituting this value into our derived interval formula:

$$\left[ 0, \frac{3.84}{n + 3.84} \right]$$

This confirms the specified interval for when  $\alpha = 0.05$ .