STA 6352, Report 7.2

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Problem

Mixtures of independent distributions: suppose the distribution of $\theta = (\theta_1, \ldots, \theta_J)$ can be written as a mixture of independent and identically distributed components:

$$p(\theta) = \int \prod_{j=1}^{J} p(\theta \mid \phi) p(\phi) \, d\phi.$$

Prove that the covariances $\mathrm{Cov}(\theta_i, \theta_j)$ are all nonnegative.

We'd like to show that the pairwise covariance is always non-negative. Since the parameters θ are exchangeable, it is sufficient to show that θ_1 and θ_2 have non-negative covariance. Using the law of total covariance, the fact that independent variables have zero covariance, and the fact that exchangeable variables have the same expectation, it follows that[1]

$$\begin{aligned}
\text{Cov}(\theta_1, \theta_2) &= \mathbb{E}\left[\text{Cov}(\theta_1, \theta_2|\phi)\right] + \text{Cov}\left(\mathbb{E}(\theta_1|\phi), \mathbb{E}(\theta_2|\phi)\right) \\
&= 0 + \mathbb{E}\left[\text{Cov}(\theta_1|\phi, \theta_2|\phi)\right] \\
&= \mathbb{E}\left[\mathbb{E}(\theta_1|\phi)\mathbb{E}(\theta_2|\phi)\right] - \mathbb{E}\left[\mathbb{E}(\theta_1|\phi)\right]\mathbb{E}\left[\mathbb{E}(\theta_2|\phi)\right] \\
&= \mathbb{E}\left[\left(\mathbb{E}(\theta_1|\phi)\right)^2\right] - \left(\mathbb{E}\left[\mathbb{E}(\theta_1|\phi)\right]\right)^2 \\
&= \text{Var}\left(\mathbb{E}(\theta_1|\phi)\right) \geq 0.
\end{aligned}$$

[1] I found this result here.