STA 6352, Report 6.5

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Problem Setup

We are given a single observation $x \sim N(\mu_1 + \mu_2, 1)$, and prior distributions $\mu_i \sim N(\varphi_i, \tau_i^2)$ for $i = 1, 2$. We are also given that μ_1 and μ_2 are independent, and we define $\mu = \mu_1 + \mu_2$.

1. Constructing Full Conditional Distributions

We want to derive $\pi(\mu_1|\mu_2, x)$ and $\pi(\mu_2|\mu_1, x)$.

We have:

$$\pi(\mu_1, \mu_2|x) \propto \pi(x|\mu_1, \mu_2)\pi(\mu_1)\pi(\mu_2)$$

Given $x = \mu_1 + \mu_2 + \epsilon$, where $\epsilon \sim N(0, 1)$, we have:

$$\pi(x|\mu_1, \mu_2) \propto \exp\left(-\frac{1}{2}(x - \mu_1 - \mu_2)^2\right)$$

Also, $\pi(\mu_i) \propto \exp\left(-\frac{1}{2\tau_i^2}(\mu_i - \varphi_i)^2\right)$ for $i = 1, 2$.

Then,

$$\pi(\mu_1, \mu_2|x) \propto \exp\left(-\frac{1}{2}(x - \mu_1 - \mu_2)^2 - \frac{1}{2\tau_1^2}(\mu_1 - \varphi_1)^2 - \frac{1}{2\tau_2^2}(\mu_2 - \varphi_2)^2\right)$$

To find $\pi(\mu_1|\mu_2, x)$, we only consider terms involving μ_1:

$$\pi(\mu_1|\mu_2, x) \propto \exp\left(-\frac{1}{2}(x - \mu_1 - \mu_2)^2 - \frac{1}{2\tau_1^2}(\mu_1 - \varphi_1)^2\right)$$

Completing the square, we find that $\pi(\mu_1|\mu_2, x)$ is a normal distribution with mean and variance:

$$\mu_{1|2,x} = \frac{\tau_1^2(x - \mu_2) + \varphi_1}{1 + \tau_1^2}$$

$$\sigma^2_{1|2,x} = \frac{\tau_1^2}{1 + \tau_1^2}$$

Similarly,

$$\pi(\mu_2|\mu_1, x) \propto \exp\left(-\frac{1}{2}(x - \mu_1 - \mu_2)^2 - \frac{1}{2\tau_2^2}(\mu_2 - \varphi_2)^2\right)$$

Completing the square, we find that $\pi(\mu_2|\mu_1, x)$ is a normal distribution with mean and variance:

$$\mu_{2|1,x} = \frac{\tau_2^2(x - \mu_1) + \varphi_2}{1 + \tau_2^2}$$

$$\sigma_{2|1,x}^2 = \frac{\tau_2^2}{1 + \tau_2^2}$$

```
gibbs_sampler <- function(x, phi1, phi2, tau1, tau2, n_iter, mu1_init, mu2_in
  mu1_samples <- numeric(n_iter)
  mu2_samples <- numeric(n_iter)
  mu_samples <- numeric(n_iter)

  mu1 <- mu1_init
  mu2 <- mu2_init

  for (i in 1:n_iter) {
    # Sample mu1
    mu1_mean <- (tau12 * (x - mu2) + phi1) / (1 + tau12)
    mu1_sd <- sqrt(tau12 / (1 + tau12))
    mu1 <- rnorm(1, mu1_mean, mu1_sd)

    # Sample mu2
    mu2_mean <- (tau22 * (x - mu1) + phi2) / (1 + tau22)
    mu2_sd <- sqrt(tau22 / (1 + tau22))
    mu2 <- rnorm(1, mu2_mean, mu2_sd)

    # Store samples
    mu1_samples[i] <- mu1
    mu2_samples[i] <- mu2
    mu_samples[i] <- mu1 + mu2
  }

  return(list(mu1 = mu1_samples, mu2 = mu2_samples, mu = mu_samples))
}
```

2. Marginal Posterior Distributions

Since the joint posterior of μ_1, μ_2 is bivariate normal, we marginalize out each parameter:

$$\mu|x \sim N\left(\frac{\phi_1 + \phi_2 + x}{3}, \frac{1}{3}\right).$$

Thus, the marginal posterior distributions are:

$$\mu_1|x \sim N\left(\frac{\phi_1 + x}{2}, \frac{1}{2 + 1/\tau_1^2}\right),$$

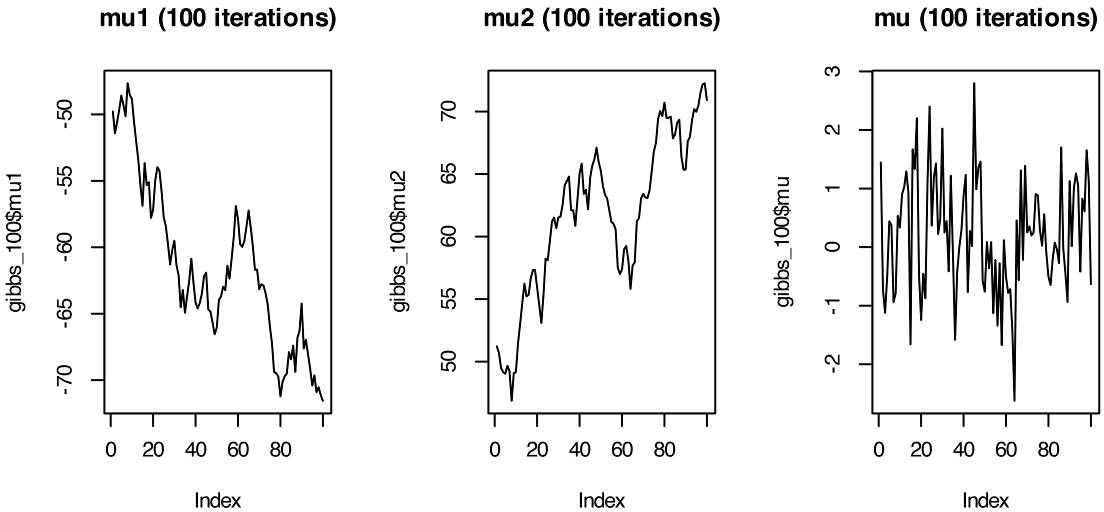
$$\mu_2|x \sim N\left(\frac{\phi_2 + x}{2}, \frac{1}{2 + 1/\tau_2^2}\right).$$

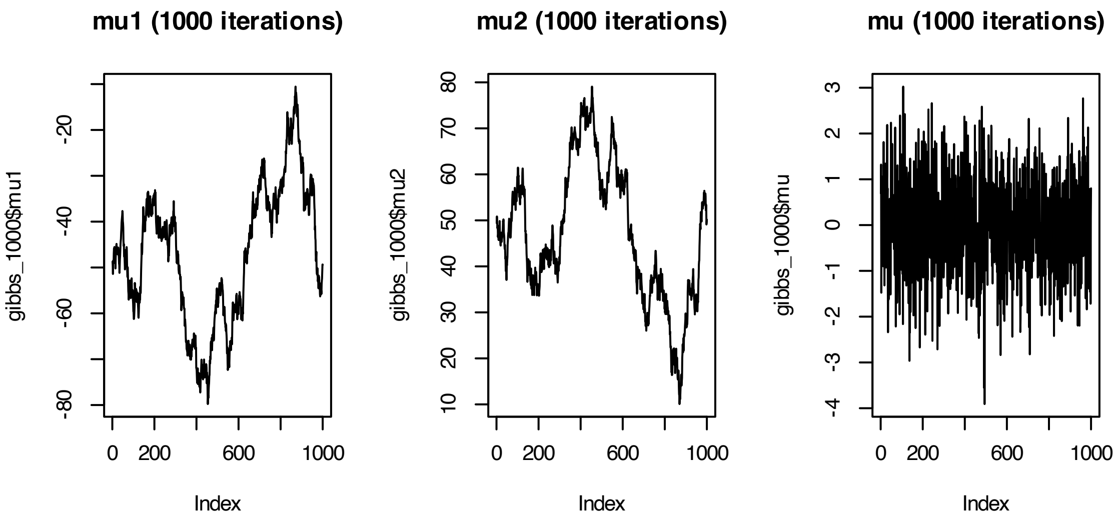
3. Running the Gibbs Sampler with $\varphi_1 = \varphi_2 = 50$, $\tau_1 = \tau_2 = 1000$, $x = 0$

```
phi1 <- 50
phi2 <- 50
tau1 <- 1000
tau2 <- 1000
x <- 0
n_iter_100 <- 100
n_iter_1000 <- 1000

# Run for 100 iterations
gibbs_100 <- gibbs_sampler(x, phi1, phi2, tau1, tau2, n_iter_100, phi1, phi2)

# Run for 1000 iterations
gibbs_1000 <- gibbs_sampler(x, phi1, phi2, tau1, tau2, n_iter_1000, phi1, phi
```





For 100 iterations, the chains do not appear to have converged. For 1000 iterations, the chains look more stable, suggesting better convergence.

Do the Data Update the Prior?

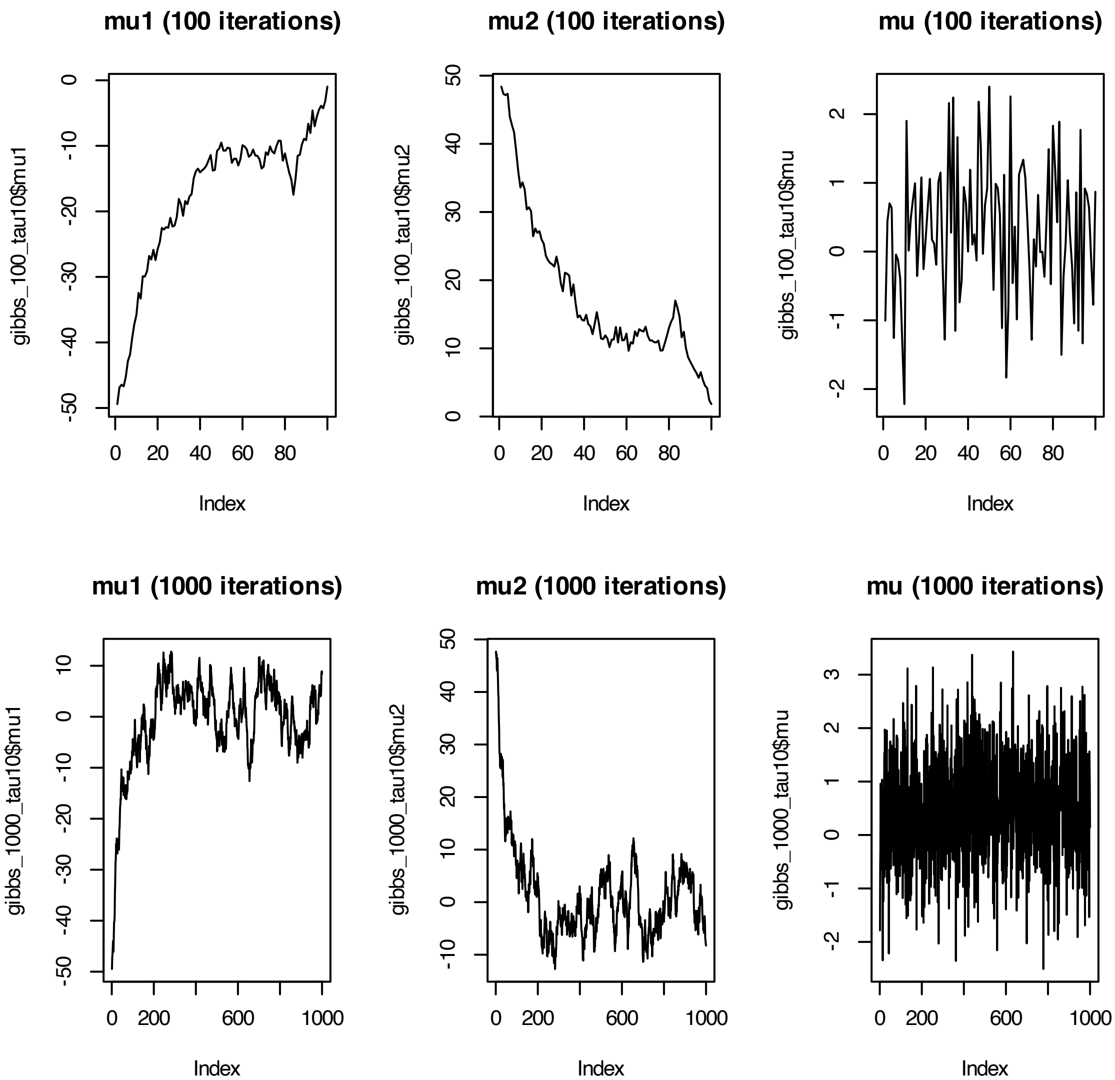
Yes. The posterior means shift from the prior means ϕ_1, ϕ_2 towards x, reflecting Bayesian updating. The posterior variances decrease compared to prior variances, indicating increased confidence in our estimates after incorporating the data.

4. Running the Gibbs Sampler with $\varphi_1 = \varphi_2 = 50$, $\tau_1 = \tau_2 = 10$, $x = 0$

```r
tau1 <- 10
tau2 <- 10

# Run for 100 iterations
gibbs_100_tau10 <- gibbs_sampler(x, phi1, phi2, tau1, tau2, n_iter_100, phi1,

# Run for 1000 iterations
gibbs_1000_tau10 <- gibbs_sampler(x, phi1, phi2, tau1, tau2, n_iter_1000, phi
```


mu1 (100 iterations)
gibbs_100_tau10$mu1
0
-10
-20
-30
-40
-50
0 20 40 60 80
Index
mu2 (100 iterations)
gibbs_100_tau10$mu2
50
40
30
20
10
0
0 20 40 60 80
Index
mu (100 iterations)
gibbs_100_tau10$mu
2
1
0
-1
-2
0 20 40 60 80
Index
mu1 (1000 iterations)
gibbs_1000_tau10$mu1
10
0
-10
-20
-30
-40
-50
0 200 600 1000
Index
mu2 (1000 iterations)
gibbs_1000_tau10$mu2
50
40
30
20
10
0
-10
0 200 600 1000
Index
mu (1000 iterations)
gibbs_1000_tau10$mu
3
2
1
0
-1
-2
0 200 600 1000
Index