STA 6360, Report 4.15

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Problem

Suppose $Y \sim \mathrm{binomial}(n, p)$.

1

Derive an exact expression for the entropy for this binomial.

Entropy of a Binomial Random Variable

Let $Y \sim \mathrm{Binomial}(n, p)$, where Y takes on values $y \in \{0, 1, \dots, n\}$ with probability mass function

$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}.$$

The entropy of Y is given by

$$H(Y) = -\sum_{y=0}^{n} P(Y = y) \log P(Y = y).$$

Substituting the probability mass function $P(Y = y)$, we get

$$H(Y) = -\sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} \log\left(\binom{n}{y} p^y (1-p)^{n-y}\right).$$

Expanding the logarithm, we use the fact that

$$\log\left(\binom{n}{y} p^y (1-p)^{n-y}\right) = \log\binom{n}{y} + y\log p + (n-y)\log(1-p),$$

to write

$$H(Y) = -\sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} \left[\log\binom{n}{y} + y\log p + (n-y)\log(1-p)\right].$$

Distributing the sum, we have

$$H(Y) = -\sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} \log\binom{n}{y} - \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} y \log p - \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} (n-y)\log(1-p).$$

Analyzing each term: 1. The first term involves $\log\binom{n}{y}$ weighted by $P(Y = y)$, which must be evaluated numerically in general. 2. The second term simplifies because $\sum_{y=0}^{n} y\binom{n}{y} p^y (1-p)^{n-y} = \mathbb{E}[Y] = np$, leading to

$$-\sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} y \log p = -np\log p.$$

3. The third term uses $\sum_{y=0}^{n}(n-y)\binom{n}{y}p^y(1-p)^{n-y} = n - \mathbb{E}[Y] = n(1-p)$, leading to

$$-\sum_{y=0}^{n}\binom{n}{y}p^y(1-p)^{n-y}(n-y)\log(1-p) = -n(1-p)\log(1-p).$$

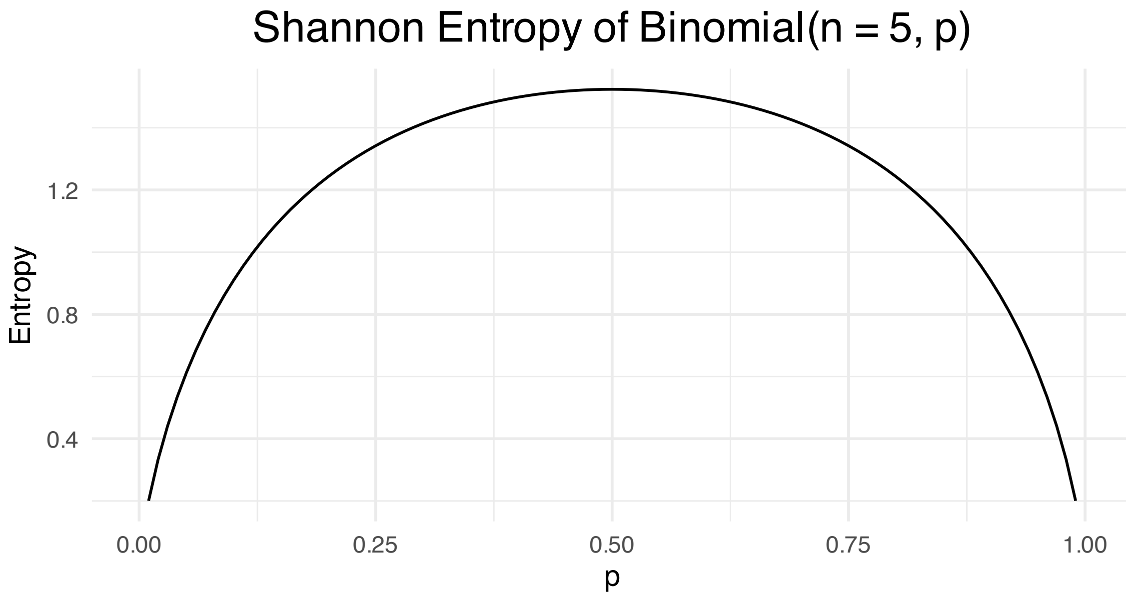
Putting it all together, the entropy becomes

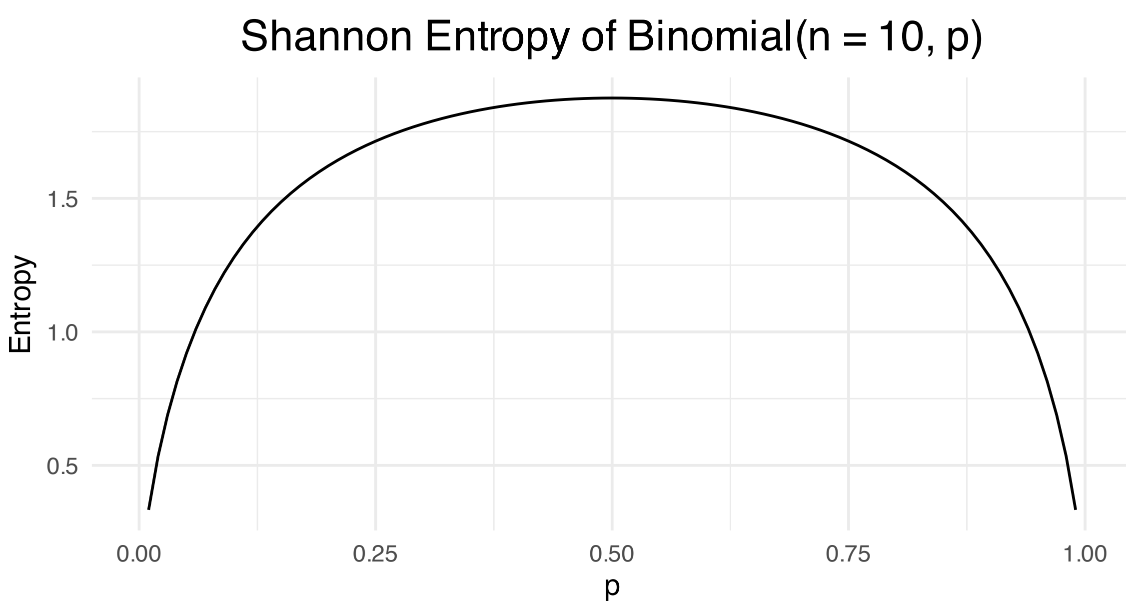
$$H(Y) = -\sum_{y=0}^{n}\binom{n}{y}p^y(1-p)^{n-y}\log\binom{n}{y} - np\log p - n(1-p)\log(1-p).$$

This is the exact expression for the entropy of a binomial random variable.

2

Using your answer in part 1, examine the entropy as a function of $p \in [0,1]$ for fixed $n = 5, 10$.



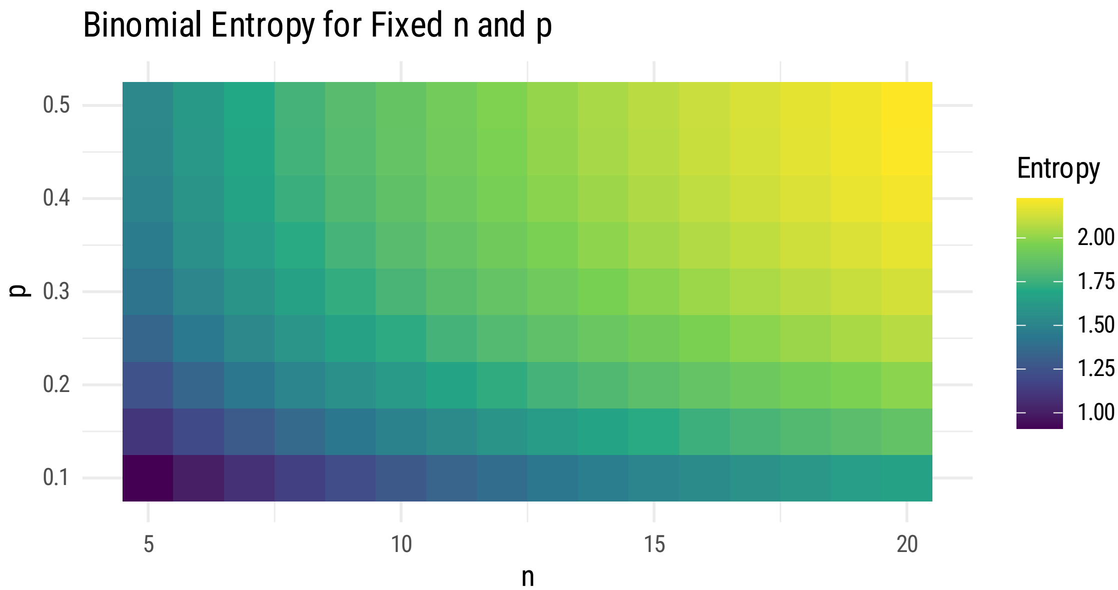


As n increases, the entropy becomes more convex.

3

Using your answer in part 1, examine the entropy as a function of $n = 5, 10, 20$ for fixed $p = 0.1, 0.25, 0.5$.

I took the liberty to explore more than the provided fixed values of n and p for this analysis.



4

Using the de-Moivre-Laplace theorem, write a large-sample approximation for the entropy derived in part 1. Compare your approximation to the exact entropy for combinations of $p = 0.1, 0.25, 0.5$ and $n = 5, 10, 20$.

Large-Sample Approximation for the Entropy of a Binomial Random Variable

Let $Y \sim \text{Binomial}(n, p)$. The probability mass function is

$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}, \quad y = 0, 1, \ldots, n.$$

The entropy of Y is given by:

$$H(Y) = -\sum_{y=0}^{n} P(Y = y) \log P(Y = y).$$

Using the de Moivre-Laplace theorem, for large n, the binomial distribution can be approximated by a normal distribution:

$$Y \sim \mathcal{N}(\mu, \sigma^2), \quad \text{where } \mu = np \text{ and } \sigma^2 = np(1-p).$$

The probability $P(Y = y)$ can be approximated by the normal density:

$$P(Y = y) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right).$$

Substituting this approximation into the entropy formula, we write:

$$H(Y) \approx -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\right) dy.$$

Expanding the logarithm:

$$\log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)\right) = -\log\sqrt{2\pi\sigma^2} - \frac{(y-\mu)^2}{2\sigma^2}.$$

Substitute this into the integral:

$$H(Y) \approx -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) \left[-\log\sqrt{2\pi\sigma^2} - \frac{(y-\mu)^2}{2\sigma^2}\right] dy.$$

Split the integral into two terms:

$$H(Y) \approx \log\sqrt{2\pi\sigma^2} + \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} \frac{(y-\mu)^2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy.$$

The second term involves the expectation of $(y-\mu)^2$ under the normal distribution, which equals σ^2. Therefore:

$$H(Y) \approx \log\sqrt{2\pi\sigma^2} + \frac{\sigma^2}{2\sigma^2}.$$

Simplify:

$$H(Y) \approx \log\sqrt{2\pi\sigma^2} + \frac{1}{2}.$$

Substitute $\sigma^2 = np(1-p)$:

$$H(Y) \approx \log\sqrt{2\pi e\, np(1-p)}.$$

Thus, the large-sample approximation for the entropy of a binomial random variable is:

$$H(Y) \approx \frac{1}{2}\log(2\pi e\, np(1-p)).$$