STA 6360, Report 1.6

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Example 1.12

Suppose we observe $y \sim \mathcal{N}(\mu, \sigma^2)$, σ^2 known, and use prior $\mu \sim \mathcal{N}(\xi, \tau^2)$ for $\mu \neq 0$ and $\rho = 1/2$. We want to test $H_0 : \mu = 0$ vs. $H_1 : \mu \neq 0$. In the absence of prior information to the contrary, we set $\xi = 0$. Then, we have it that our prior is

$$\pi(\mu) = \rho \; I_{\{\mu=0\}}(\mu) + \rho \, \pi_1(\mu) \, I_{\{\mu \neq 0\}}.$$

Under H_1, the prior for μ is $\mu \sim \mathcal{N}(0, \tau^2)$. Thus, the posterior distribution of y given μ and H_1 is $y \mid \mu, H_1 \sim \mathcal{N}(\mu, \sigma^2)$. To obtain the marginal likelihood $m(y \mid H_1)$, we integrate out μ with respect to its prior:

$$m(y \mid H_1) = \int_{-\infty}^{\infty} f(y \mid \mu) \, \pi_1(\mu) \, d\mu.$$

Since $y \mid \mu \sim \mathcal{N}(\mu, \sigma^2)$ and $\mu \sim \mathcal{N}(0, \tau^2)$, y marginally follows a normal distribution with mean 0 and variance $\sigma^2 + \tau^2$. Therefore, $y \mid H_1 \sim \mathcal{N}(0, \sigma^2 + \tau^2)$, and thus,

$$m(y \mid H_1) = \frac{1}{\sqrt{2\pi(\sigma^2 + \tau^2)}} \exp\left(-\frac{y^2}{2(\sigma^2 + \tau^2)}\right).$$

The Bayes factor BF against H_0 is given by

$$BF = \frac{m(y \mid H_1)}{m(y \mid H_0)}.$$

Substitute the expressions for $m(y \mid H_1)$ and $m(y \mid H_0)$:

$$BF = \frac{\frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}} \exp\left\{-\frac{y^2}{2(\sigma^2+\tau^2)}\right\}}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{y^2}{2\sigma^2}\right\}}.$$

Simplify this expression yields

$$BF = \sqrt{\frac{\sigma^2}{\sigma^2 + \tau^2}} \exp\left\{\frac{y^2}{2}\left(\frac{1}{\sigma^2 + \tau^2} - \frac{1}{\sigma^2}\right)\right\}.$$

Rearrange the exponent term, we have

$$BF = \sqrt{\frac{\sigma^2}{\sigma^2 + \tau^2}} \exp\left\{\frac{\tau^2 y^2}{2\sigma^2(\sigma^2 + \tau^2)}\right\}.$$

Then, we compute the posterior probability. To calculate the posterior probability of the null hypothesis, $H_0 : \mu = 0$, we use Bayes' theorem for model probabilities. Given prior probabilities $P(H_0) = \rho$ and $P(H_1) = 1 - \rho$, the posterior probability of H_0 is

$$P(H_0 \mid y) = \frac{P(H_0) \, m(y \mid H_0)}{P(H_0) \, m(y \mid H_0) + P(H_1) \, m(y \mid H_1)},$$

where:

- $m(y \mid H_0)$ and $m(y \mid H_1)$ are the marginal likelihoods under H_0 and H_1, respectively.
- BF is the Bayes factor in favor of H_1 over H_0.

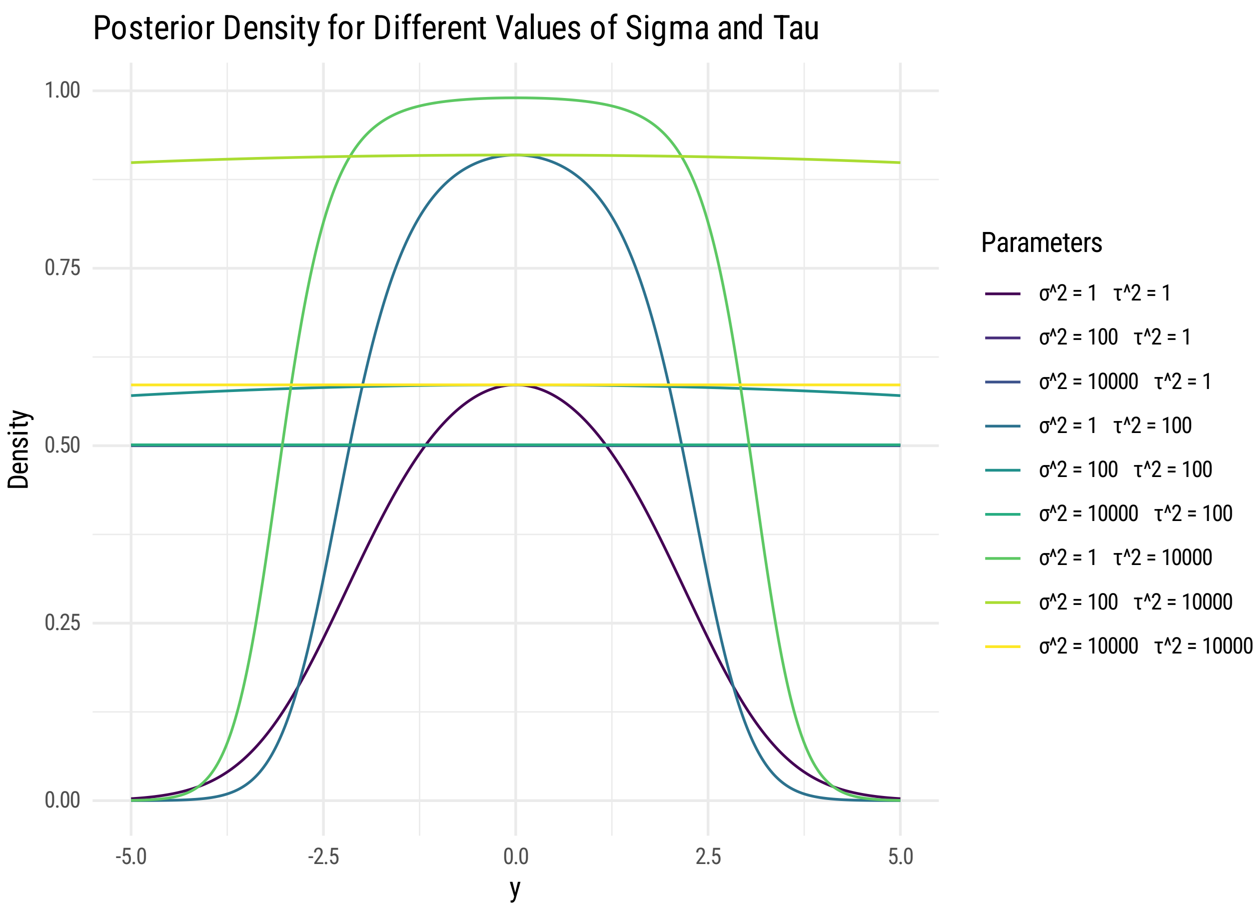
Since $m(y \mid H_1) = BF \cdot m(y \mid H_0)$, we can rewrite the posterior probability of H_0 as:

$$P(H_0 \mid y) = \frac{\rho}{\rho + (1-\rho)BF}.$$

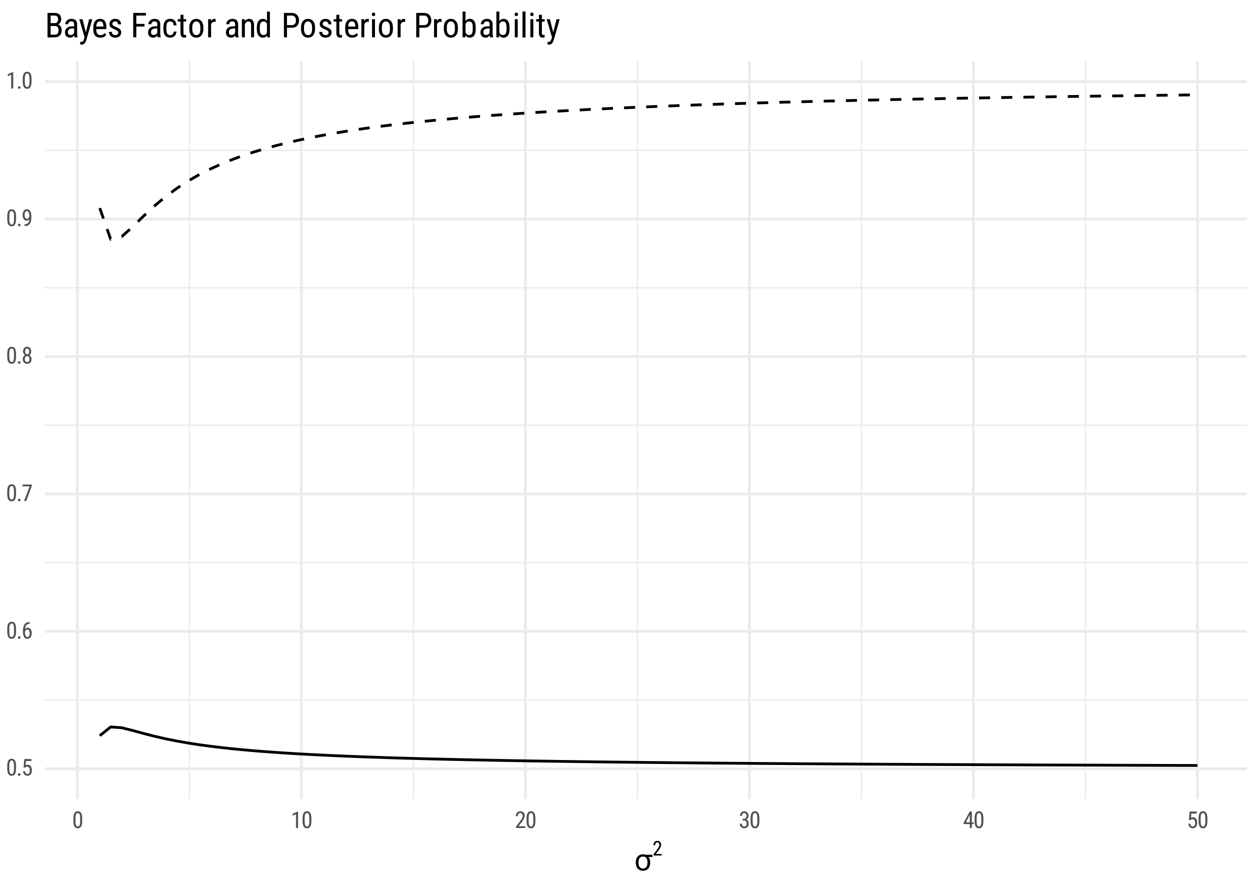
Substituting the Bayes factor we derived, we obtain the posterior probability of H_0:

$$P(H_0 \mid y) = \left[1 + \frac{1-\rho}{\rho}\sqrt{\frac{\sigma^2}{\sigma^2+\tau^2}}\exp\left\{\frac{\tau^2 y^2}{2\sigma^2(\sigma^2+\tau^2)}\right\}\right]^{-1}.$$

Assessing Relationship



Other Investigation



Bayes factor (dashed) and posterior probability (dotted) as variance increases.