STA 6360, Report 1.5

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Example 10.1 (p. 272-273 Rosner et. al)

Suppose we have a case where we have a set of $n = 404$ Bernoulli trials, with $y = 184$ successes. We are wanting to test the hypotheses $H_0 : \theta \leq 0.5$ against $H_1 : \theta > 0.5$. We impose a $\text{Unif}(0, 1)$ prior on θ. Then, to compute the Bayes factor, we have that since $q_0 = q_1 = 0.5$,

$$\begin{aligned} BF_{10} &= \frac{\Pr(M_1|\mathbf{x})}{\Pr(M_0|\mathbf{x})} \\ &= \frac{\int_{0.5}^{1} \theta^{1+184-1}(1-\theta)^{1+404-184-1} d\theta}{\int_{0}^{0.5} \theta^{1+184-1}(1-\theta)^{1+404-184-1} d\theta} \\ &\approx 0.0382. \end{aligned}$$

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## [1] 0.03815624
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Given, the Bayes factor is 0.0382, this would indicate that the plausibility of the data under M_0 is 26 times what it is under M_1. Now, if the prior odds are equal to 1, we would thus likely accept the hypothesis that the proportion of veterans with uncontrolled hypertension is less than 0.5.

Exercise 10.1

Let $\{Y_i : \ i = 1, \ldots, n\}$ be from $M_0 : Y_i \overset{iid}{\sim} \mathcal{N}(0, 1/\tau)$ or $M_1 : Y_i \overset{iid}{\sim} \mathcal{N}(\mu, 1/\tau)$. Let $p_i(\tau) \sim \text{IG}(\alpha, \beta)$ and $p_1(\mu) \sim \mathcal{N}(\mu_0, \sigma_0^2)$. Define $\theta_0 = \tau$ and $\theta_1 = (\mu, \tau)'$.

(a)

Obtain explicit formulas for the prior predictive distribution of Y and the Bayes Factor. Simpliy the formula for the Bayes Factor.

Under M_0:

The prior predictive distribution under M_0 is:

$$p(Y_i \mid M_0) = \int_0^\infty \mathcal{N}(Y_i \mid 0, 1/\tau) \, \text{IG}(\tau \mid \alpha, \beta) \, d\tau = t_{2\alpha}(0, \beta/\alpha).$$

Thus,

$$Y_i \mid M_0 \sim t_{2\alpha}(0, \beta/\alpha).$$

Under M_1:

We want to compute the prior predictive distribution under model M_1:

$$p(Y_i \mid M_1) = \int_0^\infty \int_{-\infty}^\infty \mathcal{N}(Y_i \mid \mu, 1/\tau) \mathcal{N}(\mu \mid \mu_0, \sigma_0^2) \, \text{IG}(\tau \mid \alpha, \beta) \, d\mu \, d\tau.$$

Substituting the likelihood and priors:

$$p(Y_i \mid M_1) = \int_0^\infty \int_{-\infty}^\infty \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{\tau(Y_i-\mu)^2}{2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{-\alpha-1} \exp\left(-\frac{\beta}{\tau}\right) d\mu\, d\tau.$$

First, integrate out μ:

$$\int_{-\infty}^\infty \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{\tau(Y_i-\mu)^2}{2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) d\mu = \sqrt{\frac{\tau}{2\pi(1+\tau\sigma_0^2)}} \exp\left(-\frac{\tau(Y_i-\mu_0)^2}{2(1+\tau\sigma_0^2)}\right).$$

Now, integrate out τ:

$$p(Y_i \mid M_1) = \int_0^\infty \sqrt{\frac{\tau}{2\pi(1+\tau\sigma_0^2)}} \exp\left(-\frac{\tau(Y_i-\mu_0)^2}{2(1+\tau\sigma_0^2)}\right) \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{-\alpha-1} \exp\left(-\frac{\beta}{\tau}\right) d\tau.$$

The result of this integral is a Student-t distribution:

$$p(Y_i \mid M_1) = t_{2\alpha}\left(\mu_0, \frac{\beta}{\alpha} + \sigma_0^2\right).$$

Thus, the prior predictive distribution of Y_i under M_1 is a Student-t distribution with 2α degrees of freedom, location μ_0, and scale $\sqrt{\frac{\beta}{\alpha} + \sigma_0^2}$.

Bayes Factor

The Bayes Factor comparing M_1 to M_0 is:

$$\mathrm{BF}_{10} = \frac{p(Y_i \mid M_1)}{p(Y_i \mid M_0)} = \frac{t_{2\alpha}(Y_i \mid \mu_0, \beta/\alpha + \sigma_0^2)}{t_{2\alpha}(Y_i \mid 0, \beta/\alpha)}.$$

(b)

[[1]] [1] “Bayes Factor for alpha = 1, beta = 1 and sigma^2_0 = 100: 0.804078635385331”
[[2]] [1] “Bayes Factor for alpha = 0.1, beta = 0.1 and sigma^2_0 = 100: 0.798463676722325”
[[3]] [1] “Bayes Factor for alpha = 0.001, beta = 0.001 and sigma^2_0 = 100: 0.797727283013699”
[[4]] [1] “Bayes Factor for alpha = 1, beta = 1 and sigma^2_0 = 1000: 0.829163998229798”
[[5]] [1] “Bayes Factor for alpha = 0.1, beta = 0.1 and sigma^2_0 = 1000: 0.823324564855323”
[[6]] [1] “Bayes Factor for alpha = 0.001, beta = 0.001 and sigma^2_0 = 1000: 0.822558804742928”
[[7]] [1] “Bayes Factor for alpha = 1, beta = 1 and sigma^2_0 = 1e+06: 0.832080715983891”
[[8]] [1] “Bayes Factor for alpha = 0.1, beta = 0.1 and sigma^2_0 = 1e+06: 0.826214915178764”
[[9]] [1] “Bayes Factor for alpha = 0.001, beta = 0.001 and sigma^2_0 = 1e+06: 0.825445706093437”