STA 6360, Report 1.1

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1.1.1

We were told to skip 1.1.1.

1.1.2

Here we have that we observe $s_n = 5$ for $n = 20$. Given the four priors $\theta \sim \text{Beta}(1,1)$, $\theta \sim \text{Beta}(2,2)$, $\theta \sim \text{Beta}(5,5)$, and $\theta \sim \text{Beta}(10,10)$. We know that the posterior, given a $\text{Beta}(a,b)$ prior is $\theta \mid \mathbf{x} \sim \text{Beta}(5+a,\ 15+b)$. Therefore for a $\text{Beta}(1,1)$ prior, we have a $\text{Beta}(6,16)$ posterior. So

$$\Pr(\theta > 0.5 | s_n = 5) \approx 0.0133.$$

For a $\text{Beta}(2,2)$ prior, we have a $\text{Beta}(7,17)$ posterior. So

$$\Pr(\theta > 0.5 | s_n = 5) \approx 0.0173.$$

For a $\text{Beta}(5,5)$ prior, we have a $\text{Beta}(10,20)$ posterior. So

$$\Pr(\theta > 0.5 | s_n = 5) \approx 0.0307.$$

For a $\text{Beta}(10,10)$ prior, we have a $\text{Beta}(15,25)$ posterior. So

$$\Pr(\theta > 0.5 | s_n = 5) \approx 0.0540.$$

Such are the respective probabilities.

1.1.3

We can use numerical integration for computing the posterior.

```
## [[1]]
## [1] "Beta(1,1) Prior: 0.0133018493652344"
##
## [[2]]
## [1] "Beta(2,2) Prior: 0.0173448324203491"
##
## [[3]]
## [1] "Beta(5,5) Prior: 0.0307141728699208"
##
## [[4]]
## [1] "Beta(10,10) Prior: 0.0540645107030286"
```

1.1.4

A frequentist approach here would rely on a normal approximation of the binomial, which is not ideal since $n = 20$. We would then say that the sampling distribution of $\theta \sim \mathcal{N}(\theta, \frac{\theta(1-\theta)}{n})$. Under the null hypothesis, $H_0: \ \theta = 0.5$. Applying this normal approximation for the binomial, we would then say that the test statistic

$$\sqrt{n}\frac{\theta - \theta_0}{\sqrt{\theta_0(1-\theta_0)}} \approx \mathcal{N}(0,\ 1).$$

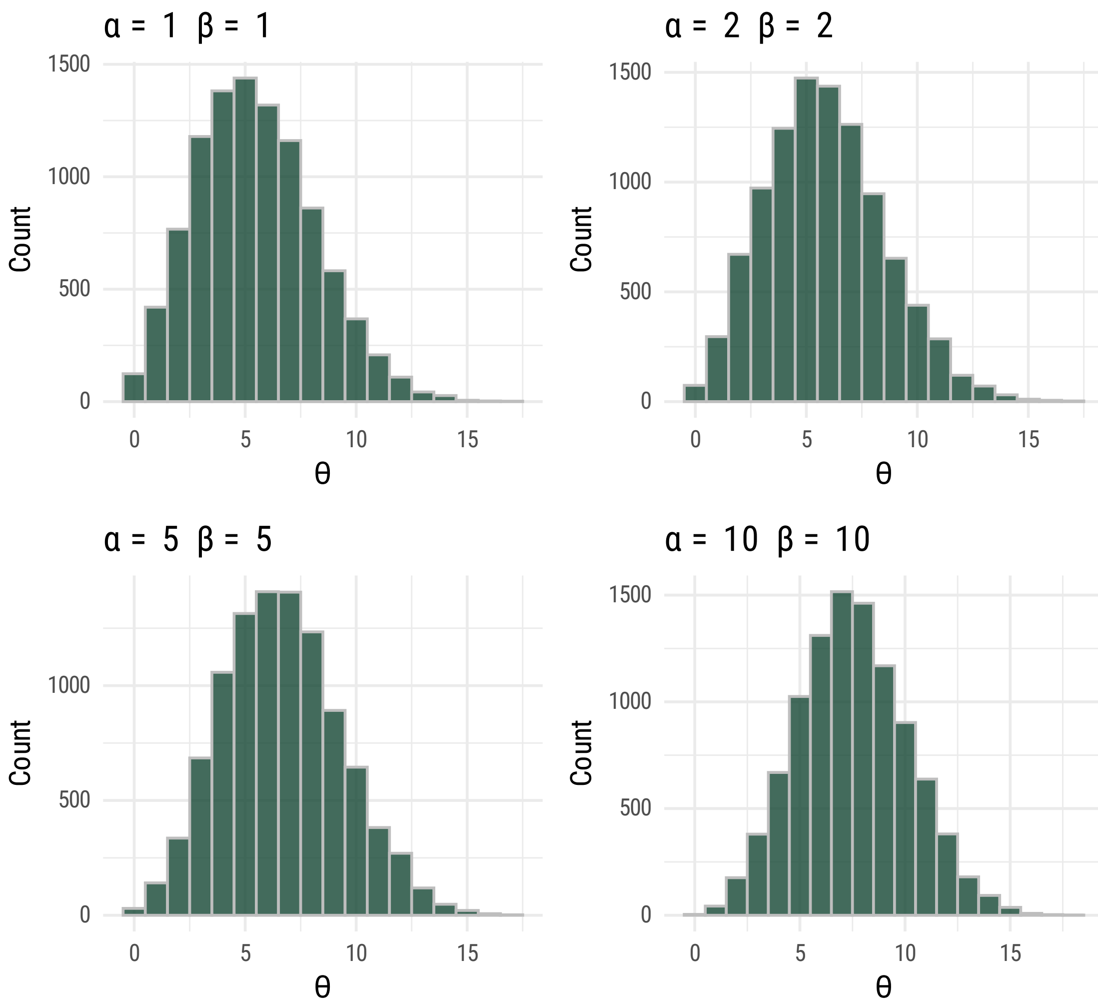
Hence, the test statistic for this would be

$$z = \sqrt{20}\left(\frac{0.25 - 0.5}{0.5}\right) = -2.236.$$

So for a critical value of $\alpha = 0.05$, we have that to test the null hypothesis against the alternative, $H_1: \theta > 0.5$, we would find $P_\theta(Z > z) \approx 0.9873$. This would lend to fail to reject the null hypothesis at the $\alpha = 0.05$ level. Compared to the posterior predictive distributions, this test seems to be much less informed.

1.1.5

```
## Loading required package: coda

## Linked to JAGS 4.3.2

## Loaded modules: basemod,bugs

## This is bayesplot version 1.11.1

## - Online documentation and vignettes at mc-stan.org/bayesplot

## - bayesplot theme set to bayesplot::theme_default()

##    * Does _not_ affect other ggplot2 plots

##    * See ?bayesplot_theme_set for details on theme setting
```



1.1.6

Finding the posterior predictive probabilites, we have that for the

Prior Shape Values	Posterior Pred. Prob.
1	0.5387
2	0.4830
5	0.3494
10	0.2336

1.1.7

From the notes, we have that the closed form for the posterior predictive probability is

$$\Pr(s_n \leq 5|s_0, f_0) = \sum_{s_n=0}^{5} \binom{n}{s_n} \frac{B(s_n + s_0 + a, f_n + f_0 + b)}{B(s_0 + a, f_0 + b)}.$$

So for $s_0 = 5$ and $f_n = 15$, the posterior predictive probability with a $\mathrm{Beta}(1, 1)$ prior would be

$$\Pr(s_n \leq 5|s_0 = 5, f_0 = 15) = \sum_{s_n=0}^{5} \binom{20}{s_n} \frac{B(s_n + 6, (20 - s_n) + 16)}{B(6, 16)} \approx 0.526.$$

The posterior predictive probability with a $\mathrm{Beta}(2, 2)$ prior would be

$$\Pr(s_n \leq 5|s_0 = 5, f_0 = 15) = \sum_{s_n=0}^{5} \binom{20}{s_n} \frac{B(s_n + 7, (20 - s_n) + 17)}{B(7, 17)} \approx 0.479.$$

The posterior predictive probability with a $\mathrm{Beta}(5, 5)$ prior would be

$$\Pr(s_n \leq 5|s_0 = 5, f_0 = 15) = \sum_{s_n=0}^{5} \binom{20}{s_n} \frac{B(s_n + 10, (20 - s_n) + 20)}{B(10, 20)} \approx 0.349.$$

The posterior predictive probability with a $\mathrm{Beta}(10, 10)$ prior would be

$$\Pr(s_n \leq 5|s_0 = 5, f_0 = 15) = \sum_{s_n=0}^{5} \binom{20}{s_n} \frac{B(s_n + 15, (20 - s_n) + 25)}{B(15, 25)} \approx 0.230.$$