STA 6351, Report.2.2

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2.2

Report 2.2: Independence of functions of exponential variates

Verify that $T = X + Y$ and $U = X/(X + Y)$ are independent. Proceed as follows:

(a) Start with the joint density of X and Y,

$$f_{X,Y}(x, y) = \theta^2 e^{-\theta(x+y)} I_{[0,\infty)}(x) I_{[0,\infty)}(y).$$

where $X \sim \text{Exp}(\theta)$ and $Y \sim \text{Exp}(\theta)$ are independent.

(b) Apply the transformation $(X, Y) \mapsto (T, U)$ given by

$$T = X + Y \quad \text{and} \quad U = \frac{X}{X + Y},$$

whose inverse is

$$X = UT \quad \text{and} \quad Y = (1 - U)T.$$

The support of the new random variables is determined by the constraints $X \geq 0$ and $Y \geq 0$. Since $T = X + Y, T \geq 0$. Since $X = UT$ and $Y = (1 - U)T$, and $T \geq 0$, we require $U \geq 0$ and $1 - U \geq 0$, which implies $0 \leq U \leq 1$. The support of (T, U) is thus $\mathcal{S}_{T,U} = \{(t, u) : t \geq 0, 0 \leq u \leq 1\}$.

(c) Compute the Jacobian determinant The Jacobian matrix $\mathbf{J}$ of the inverse transformation is:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{pmatrix} = \begin{pmatrix} u & t \\ 1 - u & -t \end{pmatrix}$$

The determinant of the Jacobian is:

$$\begin{aligned} \det(\mathbf{J}) &= \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial u}\right) - \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial t}\right) \\ &= (u)(-t) - (t)(1 - u) \\ &= -ut - t + ut \\ &= -t. \end{aligned}$$

The Jacobian determinant for the density transformation is the absolute value:

$$\left|\frac{\partial(x, y)}{\partial(t, u)}\right| = |-t| = t, \quad \text{since } t \geq 0.$$

(d) Show that the joint density of (T,U) is The joint density $f_{T,U}(t,u)$ is found using the formula $f_{T,U}(t,u) = f_{X,Y}(x(t,u), y(t,u)) \cdot |\det(\mathbf{J})|$. Substitute $x = ut$ and $y = (1-u)t$ into $f_{X,Y}(x,y)$:

$$f_{X,Y}(ut, (1-u)t) = \theta^2 e^{-\theta(ut+(1-u)t)} I_{[0,\infty)}(ut) I_{[0,\infty)}((1-u)t)$$

Since $t \geq 0$ and $0 \leq u \leq 1$, $ut \geq 0$ and $(1-u)t \geq 0$. The indicator functions simplify:

$$\begin{aligned} f_{X,Y}(ut, (1-u)t) &= \theta^2 e^{-\theta(ut+t-ut)} I_{[0,\infty)}(t) I_{[0,1]}(u) \\ &= \theta^2 e^{-\theta t} I_{[0,\infty)}(t) I_{[0,1]}(u). \end{aligned}$$

Multiplying by the Jacobian determinant $|\det(\mathbf{J})| = t$:

$$\begin{aligned} f_{T,U}(t,u) &= \left(\theta^2 e^{-\theta t} I_{[0,\infty)}(t) I_{[0,1]}(u)\right) \cdot t \\ &= \theta^2 t e^{-\theta t} I_{[0,\infty)}(t) I_{[0,1]}(u). \end{aligned}$$

This can be factored into marginal densities:

$$f_{T,U}(t,u) = \underbrace{[\theta^2 t e^{-\theta t} I_{[0,\infty)}(t)]}_{f_T(t)} \underbrace{[I_{[0,1]}(u)]}_{f_U(u)}$$

Since the joint density $f_{T,U}(t,u)$ is equal to the product of the marginal densities $f_T(t) f_U(u)$, T and U are **independent**.

(e) Conclude that $T \sim \text{Gamma}(2,\theta)$ and $U \sim \text{Uniform}(0,1)$. Based on the factored marginal densities:

$$f_T(t) = \theta^2 t e^{-\theta t} I_{[0,\infty)}(t) = \frac{\theta^2 t^{2-1} e^{-\theta t}}{\Gamma(2)} I_{[0,\infty)}(t) \implies T \sim \text{Gamma}(2,\theta)$$

$$f_U(u) = I_{[0,1]}(u) = \frac{1}{1-0} I_{[0,1]}(u) \implies U \sim \text{Uniform}(0,1)$$