# STA 6351, Report.1.9

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### 1.9 Cross-checking Poisson-coupled-normal moments. Establish (1.10.4).

We have, for this example,

$$
\begin{aligned}
Y_i &\sim \text{Poisson}(\lambda) \\
X_i | Y_i = y &\sim \begin{cases} \delta_0, & y = 0, \\ N(\mu, \sigma^2), & y \geq 1 \end{cases}
\end{aligned}
$$

where $\delta_0$ denotes a pointw mass at the origin. Zeros in the model arise structurally when $Y_i = 0$; nonzeros are mixtures of normals indexed by $y \geq 1$. $\lambda$ is the expected number of electrons per window. The mean contribution to amplitude per electron (system gain) is $\mu$. Finally, $\sigma^2$ is the per-electron variance.

#### (a) Establish (1.10.4) via double-expectation arguments.

We have that in (1.10.4),

$$\mathbb{E}[X] = \lambda\mu, \quad \mathbb{V}(X) = \lambda(\mu^2 + \sigma^2)$$

We have by double-expectation:

$$
\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}_Y[\mathbb{E}[X|Y]] \\
&= \mathbb{E}_Y[Y\mu] \\
&= \mu\mathbb{E}[Y] = \mu\lambda = \lambda\mu.
\end{aligned}
$$

Then, by the Law of Total Variance:

$$
\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}_Y[\mathbb{V}(X|Y)] + \mathbb{V}_Y[\mathbb{E}[X|Y]] \\
&= \mathbb{E}_Y[Y\sigma^2] + \mathbb{V}_Y[Y\mu] \\
&= \sigma^2\mathbb{E}[Y] + \mu^2\mathbb{V}[Y] \\
&= \sigma^2\lambda + \mu^2\lambda \\
&= \lambda(\mu^2 + \sigma^2)
\end{aligned}
$$

#### (b) Establish (1.10.3).

We have that (1.10.3) is:

$$
\begin{aligned}
M_X(t) = \mathbb{E}[e^{tX}] &= \exp\left\{\lambda\left(M_Z(t) - 1\right)\right\} \\
&= \exp\left\{\lambda\left(\exp(\mu t + \frac{1}{2}\sigma^2 t^2) - 1\right)\right\}.
\end{aligned}
$$

The moment-generating function (MGF) of $X$ is established by recognizing that $X = \sum_{k=1}^{Y} Z_k$, where $Y \sim \text{Poisson}(\lambda)$ and $Z_k \overset{iid}{\sim} N(\mu, \sigma^2)$.

The general formula for the MGF of a compound Poisson distribution is:

$$M_X(t) = \exp\{\lambda\,(M_Z(t) - 1)\}$$

where $M_Z(t)$ is the MGF of the component distribution $Z \sim N(\mu, \sigma^2)$.

The MGF for $Z$ is:

$$M_Z(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

Substituting $M_Z(t)$ into the compound Poisson formula yields the required result (1.10.3):

$$\begin{aligned} M_X(t) &= \exp\{\lambda\,(M_Z(t) - 1)\} \\ &= \exp\left\{\lambda\left(\exp(\mu t + \frac{1}{2}\sigma^2 t^2) - 1\right)\right\} \end{aligned}$$

**(c) Establish (1.10.4) via the MGF.**

For the derivation of the mean, we apply the chain rule to $M_X(t)$. $M_X'(t) = M_X(t) \cdot \frac{d}{dt}\left[\lambda\,(e^{\cdots} - 1)\right]$.

$$\begin{aligned} M_X'(t) &= M_X(t) \cdot \lambda e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t) \\ \mathbb{E}[X] &= M_X'(0) \\ &= M_X(0) \cdot \lambda e^0(\mu + 0) \\ &= 1 \cdot (\lambda \cdot 1 \cdot \mu) \\ &= \lambda\mu \end{aligned}$$

Since $M_X(0) = 1$ and $e^0 = 1$, the mean simplifies to $\lambda\mu$.

For the variance derivation, we apply the product rule to $M_X'(t)$. Let $g'(t)$ be the derivative of the exponent's argument. $M_X''(t) = M_X'(t)g'(t) + M_X(t)g''(t)$.

$$\begin{aligned} M_X''(t) &= M_X'(t) \underbrace{\left[\lambda e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)\right]}_{g'(t)} + M_X(t) \underbrace{\left[\lambda e^{\mu t + \frac{1}{2}\sigma^2 t^2}\left((\mu + \sigma^2 t)^2 + \sigma^2\right)\right]}_{g''(t)} \\ M_X''(0) &= M_X'(0)g'(0) + M_X(0)g''(0) \\ &= (\lambda\mu)(\lambda\mu) + 1 \cdot \lambda\left[\mu^2 + \sigma^2\right] \\ &= \lambda^2\mu^2 + \lambda\mu^2 + \lambda\sigma^2 \\ \mathbb{V}(X) &= M_X''(0) - (\mathbb{E}[X])^2 \\ &= (\lambda^2\mu^2 + \lambda\mu^2 + \lambda\sigma^2) - (\lambda\mu)^2 \\ &= \lambda^2\mu^2 + \lambda\mu^2 + \lambda\sigma^2 - \lambda^2\mu^2 \\ &= \lambda(\mu^2 + \sigma^2) \end{aligned}$$

The $\lambda^2\mu^2$ terms cancel out, leaving the final variance $\lambda(\mu^2 + \sigma^2)$, which is $\lambda \cdot \mathbb{E}[Y^2]$.