# STA 6351, Report.1.8

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### 1.8

*Data Setup and Initial Values*

We consider the **Poisson–gated normal model** with threshold $c = 1$. Observed data ($n = 8$):

$$\begin{aligned} Y_i &: 0,2,1,3,0,1,4,0 \quad \Rightarrow \quad \sum Y_i = 11, \\ \Delta_i = 1\{Y_i \le 1\} &: 1,0,1,0,1,1,0,1 \quad \Rightarrow \quad m = \sum \Delta_i = 5, \\ X_i^{\text{obs}}\,(\Delta_i = 1) &: 2.1, 1.5, 2.8, 0.9, 2.0 \quad \Rightarrow \quad S_1 = \sum_{\Delta_i=1} X_i = 9.3. \end{aligned}$$

Starting values:

$$\begin{aligned} \mu^{(0)} &= \bar{Y} = \frac{11}{8} = 1.375, \\ (\sigma^2)^{(0)} &= 1.00. \end{aligned}$$

*Intermediate Quantities at* $\theta^{(0)} = (1.375, 1.00)$

$$\begin{aligned} S_\mu &= \sum_{\Delta_i=1} (x_i - \mu^{(0)}), \\ S_2 &= \sum_{\Delta_i=1} (x_i - \mu^{(0)})^2. \end{aligned}$$

$$\begin{aligned} S_\mu &= (2.1 - 1.375) + (1.5 - 1.375) + (2.8 - 1.375) + (0.9 - 1.375) + (2.0 - 1.375) \\ &= 2.425, \\ S_2 &= (0.725)^2 + (0.125)^2 + (1.425)^2 + (-0.475)^2 + (0.625)^2 = 3.188125. \end{aligned}$$

Poisson part check:

$$\sum_{i=1}^{n} \left( \frac{Y_i}{\mu^{(0)}} - 1 \right) = \frac{11}{1.375} - 8 = 0.$$

*Score Functions*

$$U_\mu(\mu,\sigma^2) = \sum_{i=1}^{n}\left(\frac{Y_i}{\mu}-1\right) + \sum_{\Delta_i=1}\frac{x_i-\mu}{\sigma^2}$$
$$= \sum_{i=1}^{n}\left(\frac{Y_i}{\mu}-1\right) + \frac{S_\mu}{\sigma^2},$$

$$U_{\sigma^2}(\mu,\sigma^2) = -\frac{m}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{\Delta_i=1}(x_i-\mu)^2$$
$$= -\frac{m}{2\sigma^2} + \frac{S_2}{2\sigma^4}.$$

At $\theta^{(0)}$:

$$U_\mu = 0 + \frac{2.425}{1.00} = 2.425,$$
$$U_{\sigma^2} = -\frac{5}{2(1.00)} + \frac{3.188125}{2(1.00)^2} = -0.9059375.$$

*Fisher Information*

$$\mathcal{I}_{\mu\mu} = \frac{n}{\mu} + \frac{m}{\sigma^2},$$
$$\mathcal{I}_{\sigma^2\sigma^2} = \frac{m}{2\sigma^4},$$
$$\mathcal{I}_{\mu,\sigma^2} = 0.$$

At $\theta^{(0)}$:

$$\mathcal{I}_{\mu\mu} = \frac{8}{1.375} + 5 = 10.81818,$$
$$\mathcal{I}_{\sigma^2\sigma^2} = \frac{5}{2(1)^2} = 2.5.$$

*Fisher Scoring Update*

$$\theta^{(1)} = \theta^{(0)} + \mathcal{I}(\theta^{(0)})^{-1}U(\theta^{(0)}),$$

which separates as

$$\mu^{(1)} = \mu^{(0)} + \frac{U_\mu}{\mathcal{I}_{\mu\mu}},$$
$$(\sigma^2)^{(1)} = (\sigma^2)^{(0)} + \frac{U_{\sigma^2}}{\mathcal{I}_{\sigma^2\sigma^2}}.$$

$$\mu^{(1)} = 1.375 + \frac{2.425}{10.81818} = 1.5992,$$
$$(\sigma^2)^{(1)} = 1.0000 + \frac{-0.9059375}{2.5} = 0.6376.$$

Hence,

$$\boxed{\theta^{(1)} \approx (1.5992,\ 0.6376)}.$$

*Comments*

- Because $\mathcal{I}$ is block-diagonal under expectation, the parameters update independently.
- The Poisson term contributed zero to $U_\mu$ at this start; other starting values would allocate influence differently.
- A second iteration (recomputing $S_\mu, S_2$ at the new parameters) would move very close to the MLEs.

*Appendix: Numerical Verification in R*

```r
# Data
Y <- c(0, 2, 1, 3, 0, 1, 4, 0)
X <- c(2.1, 1.5, 2.8, 0.9, 2.0)
Delta <- c(1, 0, 1, 0, 1, 1, 0, 1)

# Constants
n <- length(Y)
m <- sum(Delta)
mu0 <- 11 / 8
sig2_0 <- 1.0

# Intermediate sums
S_mu <- sum(X - mu0)
S_2 <- sum((X - mu0)2)

# Score components
U_mu <- sum(Y / mu0 - 1) + S_mu / sig2_0
U_sig2 <- -m / (2 * sig2_0) + S_2 / (2 * sig2_02)

# Fisher information
I_mu_mu <- n / mu0 + m / sig2_0
I_sig2_sig2 <- m / (2 * sig2_02)

# Updates
mu1 <- mu0 + U_mu / I_mu_mu
sig2_1 <- sig2_0 + U_sig2 / I_sig2_sig2
```

```r
# Display results
verification <- tibble(
  Quantity = c(
    "S_mu",
    "S_2",
    "U_mu",
    "U_sigma2",
    "I_mu,mu",
    "I_sigma2,sigma2",
    "mu(1)",
    "sigma2(1)"
  ),
  Value = c(S_mu, S_2, U_mu, U_sig2, I_mu_mu, I_sig2_sig2, mu1, sig2_1)
)

knitr::kable(verification, digits = 6, caption = "Verification of Calculations
```

Table 1: Verification of Calculations

| Quantity | Value |
|---|---:|
| S_mu | 2.425000 |
| S_2 | 3.188125 |
| U_mu | 2.425000 |
| U_sigmaˆ2 | -0.905938 |
| I_mu,mu | 10.818182 |
| I_sigma$^{2,\text{sigma}}$2 | 2.500000 |
| muˆ(1) | 1.599160 |
| sigma$^2$(1) | 0.637625 |