STA 6351, Report.1.3

Carson Slater *Baylor University*

1.3

Recall the life-testing experiment in Report 1.1. For that setup, do the following:

1. Compute the score function for θ given $\mathbf{t}_n$.

$$\begin{aligned} U(\theta) &= \frac{\partial \ell(\theta|\mathbf{t}_n)}{\partial \theta} \\ &= \frac{\partial}{\partial \theta}\left[-r\log\theta - \frac{1}{\theta}\left(\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right)\right] \\ &= -\frac{r}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right) \end{aligned}$$

2. Find the maximum likelihood estimator $\hat{\theta}$ and comment on its form compared to the complete-data MLE for this problem; that is, compare it to the MLE for an uncensored exponential sample of size n.

$$\text{Setting } U(\theta) = 0 :$$

$$-\frac{r}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right) = 0$$

$$\frac{1}{\theta^2}\left(\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right) = \frac{r}{\theta}$$

$$\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)} = r\theta$$

$$\hat{\theta} = \frac{1}{r}\left[\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right]$$

Comparison: For complete data with n observations, the MLE is $\hat{\theta}_{\text{complete}} = \frac{1}{n}\sum_{i=1}^{n} t_i$.

The Type II censored MLE has the same form total time on test divided by number of failures. The total time includes observed failure times plus the time the $(n-r)$ censored units were under observat

3. Show that $\hat{\theta}$ is unbiased and derive its variance. {[ChatGPT assisted]}

We define $\lambda = 1/\theta$. Let

$$W_1 = t_{(1)}, \qquad W_j = t_{(j)} - t_{(j-1)}, \quad j \geq 2.$$

For i.i.d. $\text{Exp}(\lambda)$ samples, the spacings W_j are independent and

$$W_j \sim \text{Exp}((n-j+1)\lambda),$$
$$\mathbb{E}[W_j] = \frac{\theta}{n-j+1},$$
$$\text{Var}(W_j) = \frac{\theta^2}{(n-j+1)^2}.$$

The estimator is

$$\hat{\theta} = \frac{1}{r}\left(\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right).$$

Note that

$$\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)} = \sum_{j=1}^{r}(n-j+1)W_j.$$

To show it is unbiased, we consider

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right] &= \sum_{j=1}^{r}(n-j+1)\,\mathbb{E}[W_j] \\
&= \sum_{j=1}^{r}(n-j+1)\,\frac{\theta}{n-j+1} \qquad \text{(since } \mathbb{E}[W_j] = \theta/(n-j+1)\text{)} \\
&= \sum_{j=1}^{r}\theta \\
&= r\theta.
\end{aligned}$$

Therefore,

$$\mathbb{E}[\hat{\theta}] = \frac{1}{r}\,\mathbb{E}\left[\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right] = \frac{1}{r}\cdot r\theta = \theta,$$

so $\hat{\theta}$ is unbiased.

Finding the variance,

$$\text{Var}\left(\sum_{i=1}^{r} t_{(i)} + (n-r)t_{(r)}\right) = \sum_{j=1}^{r}(n-j+1)^2\,\text{Var}(W_j),$$

because the W_j are independent.

Since $\text{Var}(W_j) = \theta^2/(n-j+1)^2$, this gives

$$\sum_{j=1}^{r}(n-j+1)^2 \cdot \frac{\theta^2}{(n-j+1)^2} = \sum_{j=1}^{r}\theta^2 = r\theta^2.$$

Thus,

$$\text{Var}(\hat{\theta}) = \frac{1}{r^2}\cdot r\theta^2 = \frac{\theta^2}{r}.$$

So $\hat{\theta}$ is unbiased and has variance θ^2/r. For a complete sample of size n, the variance would be θ^2/n. Hence, with censoring, the variance depends only on the number of observed failures r.