

STA 6351, Report.1.17

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1.17

Let (T_1, T_2) follow the bivariate Poisson construction:

$$T_1 = U + W \quad \text{and} \quad T_2 = V + W,$$

with $U \sim \text{Poisson}(\lambda_1)$, $V \sim \text{Poisson}(\lambda_2)$, $W \sim \text{Poisson}(\lambda_3)$ independent. Then

$$\begin{aligned} \mathbf{E}(T_1, T_2) &= (\theta_1, \theta_2) = (\lambda_1 + \lambda_3, \lambda_2 + \lambda_3), \\ \mathbf{V}(T_1) &= \theta_1, \quad \mathbf{V}(T_2) = \theta_2, \quad \text{and} \quad \mathbf{Cov}(T_1, T_2) = \lambda_3. \end{aligned}$$

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(a) Approximate $\mathbf{V}[g(T)]$ for $g(T_1, T_2) = \log(T_1 + T_2)$.

The delta method approximates the variance of $g(\mathbf{T})$ as $\mathbf{V}[g(\mathbf{T})] \approx [\nabla g(\boldsymbol{\theta})]^T \boldsymbol{\Sigma} [\nabla g(\boldsymbol{\theta})]$, where $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and $\boldsymbol{\Sigma} = \mathbf{V}(\mathbf{T})$. The partial derivatives of $g(T_1, T_2) = \log(T_1 + T_2)$ are:

$$\begin{aligned} \frac{\partial g}{\partial T_1} &= \frac{1}{T_1 + T_2} \\ \frac{\partial g}{\partial T_2} &= \frac{1}{T_1 + T_2} \end{aligned}$$

The gradient evaluated at the mean $\boldsymbol{\theta}$ is:

$$\nabla g(\boldsymbol{\theta}) = \begin{pmatrix} \frac{1}{\theta_1 + \theta_2} \\ \frac{1}{\theta_1 + \theta_2} \end{pmatrix} = \frac{1}{\theta_1 + \theta_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

From the problem setup, the covariance matrix is:

$$\boldsymbol{\Sigma} = \mathbf{V}(\mathbf{T}) = \begin{pmatrix} \mathbf{V}(T_1) & \mathbf{Cov}(T_1, T_2) \\ \mathbf{Cov}(T_2, T_1) & \mathbf{V}(T_2) \end{pmatrix} = \begin{pmatrix} \theta_1 & \lambda_3 \\ \lambda_3 & \theta_2 \end{pmatrix}$$

Substituting the gradient and covariance matrix into the delta method formula:

$$\begin{aligned}
\mathbf{V}[g(\mathbf{T})] &\approx \begin{pmatrix} 1 & 1 \\ \theta_1 + \theta_2 & \theta_1 + \theta_2 \end{pmatrix} \begin{pmatrix} \theta_1 & \lambda_3 \\ \lambda_3 & \theta_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\theta_1 + \theta_2} \\ \frac{1}{\theta_1 + \theta_2} \end{pmatrix} \\
&= \frac{1}{(\theta_1 + \theta_2)^2} (1 \quad 1) \begin{pmatrix} \theta_1 & \lambda_3 \\ \lambda_3 & \theta_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{(\theta_1 + \theta_2)^2} (\theta_1 + \lambda_3 \quad \lambda_3 + \theta_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{(\theta_1 + \theta_2)^2} [(\theta_1 + \lambda_3) \cdot 1 + (\lambda_3 + \theta_2) \cdot 1] \\
&= \frac{1}{(\theta_1 + \theta_2)^2} [\theta_1 + \theta_2 + 2\lambda_3]
\end{aligned}$$

(b) Compute the partial derivatives

$$g'_1(\theta) = \frac{\partial}{\partial \theta_1} g(\theta_1, \theta_2) \quad \text{and} \quad g'_2(\theta) = \frac{\partial}{\partial \theta_2} g(\theta_1, \theta_2).$$

The function is $g(\theta_1, \theta_2) = \log(\theta_1 + \theta_2)$.

$$\begin{aligned}
g'_1(\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_1} [\log(\theta_1 + \theta_2)] \\
&= \frac{1}{\theta_1 + \theta_2} \cdot \frac{\partial}{\partial \theta_1} (\theta_1 + \theta_2) \\
&= \frac{1}{\theta_1 + \theta_2} \cdot (1 + 0) \\
&= \frac{1}{\theta_1 + \theta_2} \\
g'_2(\boldsymbol{\theta}) &= \frac{\partial}{\partial \theta_2} [\log(\theta_1 + \theta_2)] \\
&= \frac{1}{\theta_1 + \theta_2} \cdot \frac{\partial}{\partial \theta_2} (\theta_1 + \theta_2) \\
&= \frac{1}{\theta_1 + \theta_2} \cdot (0 + 1) \\
&= \frac{1}{\theta_1 + \theta_2}
\end{aligned}$$

(c) Substitute into the general approximation

$$\mathbf{V}[g(T)] \approx (g'_1(\theta))^2 \mathbf{V}(T_1) + (g'_2(\theta))^2 \mathbf{V}(T_2) + 2g'_1(\theta)g'_2(\theta) \mathbf{Cov}(T_1, T_2).$$

Substituting the partial derivatives and the variance components:

$$\begin{aligned}\mathbf{V}[g(\mathbf{T})] &\approx \left(\frac{1}{\theta_1 + \theta_2}\right)^2 \theta_1 + \left(\frac{1}{\theta_1 + \theta_2}\right)^2 \theta_2 + 2 \left(\frac{1}{\theta_1 + \theta_2}\right) \left(\frac{1}{\theta_1 + \theta_2}\right) \lambda_3 \\ &= \frac{1}{(\theta_1 + \theta_2)^2} \theta_1 + \frac{1}{(\theta_1 + \theta_2)^2} \theta_2 + \frac{2}{(\theta_1 + \theta_2)^2} \lambda_3 \\ &= \frac{1}{(\theta_1 + \theta_2)^2} [\theta_1 + \theta_2 + 2\lambda_3]\end{aligned}$$

Note: This result confirms the approximation derived using matrix notation in part (a), as the scalar and matrix forms of the delta method are equivalent.

(d) Simplify the expression to show that the approximate variance depends on $\theta_1, \theta_2, \lambda_3$ only through the linear combination $\theta_1 + \theta_2$.

The approximate variance from part (c) is:

$$\mathbf{V}[g(\mathbf{T})] \approx \frac{1}{(\theta_1 + \theta_2)^2} [\theta_1 + \theta_2 + 2\lambda_3]$$

We use the relationships $\theta_1 = \lambda_1 + \lambda_3$ and $\theta_2 = \lambda_2 + \lambda_3$. The term $\theta_1 + \theta_2$ can be simplified:

$$\theta_1 + \theta_2 = (\lambda_1 + \lambda_3) + (\lambda_2 + \lambda_3) = \lambda_1 + \lambda_2 + 2\lambda_3.$$

The expression is already written such that the variance components in the numerator, θ_1, θ_2 , and λ_3 , are summed, and this entire sum is divided by a function of the linear combination $\theta_1 + \theta_2$. We can rewrite the numerator explicitly in terms of λ 's:

$$\begin{aligned}\mathbf{V}[g(\mathbf{T})] &\approx \frac{1}{(\theta_1 + \theta_2)^2} [(\lambda_1 + \lambda_3) + (\lambda_2 + \lambda_3) + 2\lambda_3] \\ &= \frac{\lambda_1 + \lambda_2 + 4\lambda_3}{(\theta_1 + \theta_2)^2}\end{aligned}$$

This expanded form clearly shows that λ_1, λ_2 , and λ_3 are all present in the numerator.

To show dependence *only* on $\theta_1 + \theta_2$, we can only isolate the $\frac{1}{(\theta_1 + \theta_2)^2}$ factor, acknowledging that the numerator contains θ_1, θ_2 , and λ_3 terms, which cannot be reduced further unless λ_3 is a function of θ_1 and θ_2 .

The final simplified form for $\mathbf{V}[g(\mathbf{T})]$ is,

$$\begin{aligned}\mathbf{V}[g(\mathbf{T})] &\approx \frac{\theta_1 + \theta_2 + 2\lambda_3}{(\theta_1 + \theta_2)^2} \\ &= \frac{1}{\theta_1 + \theta_2} + \frac{2\lambda_3}{(\theta_1 + \theta_2)^2}.\end{aligned}$$

Note: The expression shows that the approximate variance is a function of the linear combination $\theta_1 + \theta_2$ and the covariance λ_3 . It is commonly written in the first line's form to emphasize the $\frac{1}{(\theta_1 + \theta_2)^2}$ factor.

(e) How does the correlation parameter λ_3 affect the approximate variance of $\log(T_1 + T_2)$?

Hint: Start by noting

$$g'_1(\theta) = \frac{1}{\theta_1 + \theta_2} \quad \text{and} \quad g'_2(\theta) = \frac{1}{\theta_1 + \theta_2}.$$

Then the cross-term with $\text{Cov}(T_1, T_2)$ will combine with the variance terms in a simple way.

Appendix

```
knitr::opts_chunk$set(  
  dev = "cairo_pdf",  
  fig.width = 5,  
  fig.height = 5,  
  fig.align = 'center',  
  echo = FALSE,  
  message = FALSE,  
  warning = FALSE,  
  error = FALSE,  
  results = 'markup'  
)  
  
# Load required libraries  
library("tidyverse")  
library("patchwork")  
library("glue")  
library("scales", warn.conflicts = FALSE)  
library("extrafont")  
library("tinytex")  
library("knitr")  
library("tidyr")  
library("latex2exp")  
library("MASS")  
library("kableExtra")  
  
theme_set(theme_minimal(base_family = "Roboto Condensed"))  
  
conflicted::conflicts_prefer(  
  readr::col_factor(),  
  purrr::discard(),  
  dplyr::lag(),  
  readr::parse_date(),  
  kableExtra::group_rows(),  
  dplyr::select  
)
```