STA 6351, Report.1.15

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1.15

Let $X \sim$ Poisson(λ) with observed count $x \in \{0, 1, 2, \ldots\}$. The log-likelihood is

$$\ell(\lambda \mid x) \equiv x \log \lambda - \lambda - \log(x!), \quad \lambda > 0.$$

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(a)

Find the MLE $\hat{\lambda}$ when $x \geq 1$, and the second and third derivatives $\ell''(\lambda)$, $\ell'''(\lambda)$.

The first derivative is

$$\ell'(\lambda) = \frac{x}{\lambda} - 1.$$

Setting $\ell'(\lambda) = 0$ gives the MLE $\hat{\lambda} = x$.

The second and third derivatives are

$$\ell''(\lambda) = -\frac{x}{\lambda^2}, \quad \text{and} \quad \ell'''(\lambda) = \frac{2x}{\lambda^3}.$$

(b)

For $x \geq 1$, write the quadratic (second-order Taylor) approximation of $\ell(\lambda)$ around $\hat{\lambda}$:

The quadratic approximation of $\ell(\lambda)$ around $\hat{\lambda}$ is,

$$\ell_Q(\lambda) \equiv \ell(\hat{\lambda}) + \tfrac{1}{2}\,\ell''(\hat{\lambda})(\lambda - \hat{\lambda})^2.$$

The linear term vanishes because $\ell'(\hat{\lambda}) = 0$ at the MLE.

(c)

Derive a remainder bound comparing the true ℓ and ℓ_Q: for $\lambda \geq \hat{\lambda}$,

$$\left|\ell(\lambda) - \ell_Q(\lambda)\right| \leq \frac{1}{6} \max_{\xi \in [\hat{\lambda}, \lambda]} |\ell'''(\xi)| \, |\lambda - \hat{\lambda}|^3.$$

For the Poisson log-likelihood this yields the relative error bound

$$\frac{\left|\ell(\lambda) - \ell_Q(\lambda)\right|}{\frac{1}{2}|\ell''(\hat{\lambda})|(\lambda - \hat{\lambda})^2} \leq \frac{2}{3} \frac{|\lambda - \hat{\lambda}|}{\hat{\lambda}}.$$

Thus, when x is small (so $\hat{\lambda} = x$ is small), the quadratic is only reliable in a very narrow neighborhood of $\hat{\lambda}$.

(d)

Work out the special case for $x = 1$: take

$$\ell(\lambda) = \log \lambda - \lambda + \text{const}.$$

The quadratic approximation is

$$\ell_Q(\lambda) = \text{const} - \tfrac{1}{2}(\lambda - 1)^2.$$

Qualitatively, the quadratic is poor for $\lambda \ll 1$ and $\lambda \gg 1$ because the Poisson log-likelihood is strongly asymmetric and not well-approximated by a parabola far from $\hat{\lambda} = 1$.

(e)

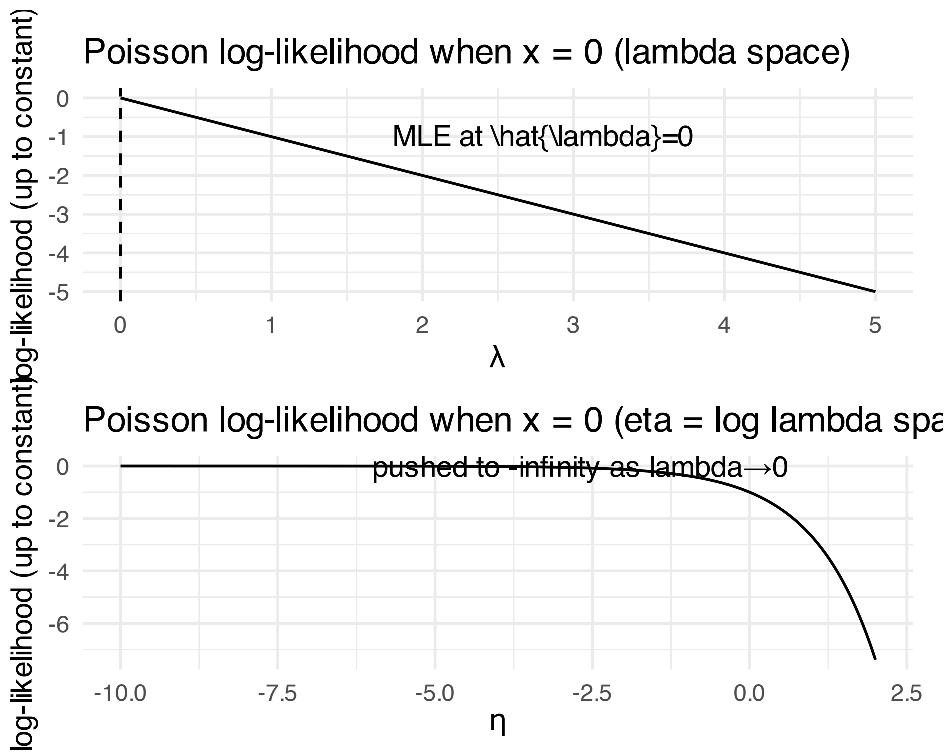
Analyze the boundary case $x = 0$.

$$\ell(\lambda \mid 0) = -\lambda.$$

This function is strictly decreasing in λ, so the MLE occurs at the boundary $\hat{\lambda} = 0$ (not in the parameter space interior).

A quadratic expansion at an interior point is unavailable—hence, the usual "log-likelihood $\approx$ parabola" heuristic completely fails.

Boundary Case: $x = 0$



This function is strictly decreasing in λ, so the MLE occurs at the boundary $\hat{\lambda} = 0$, not in the interior. Because no interior point satisfies the first-order condition, the quadratic (parabolic) approximation fails—the likelihood is linear, not curved, at the optimum.

(f)

As a remedy for the failure seen in (e), consider the reparameterization $\eta = \log \lambda$ **with** $\eta \in \mathbb{R}$**.**

$$\begin{aligned} \ell(\eta \mid x) &= x\eta - e^{\eta} - \log(x!), \\ \ell'(\eta) &= x - e^{\eta}, \\ \ell''(\eta) &= -e^{\eta}. \end{aligned}$$

For $x \geq 1$, the MLE is $\hat{\eta} = \log x$ with curvature $\ell''(\hat{\eta}) = -x$.

When x is small, the curvature is small in magnitude (the surface is flat), so even in η the quadratic can be fragile—though this reparameterization avoids the boundary non-existence seen at $x = 0$ in (e).

Appendix

```
knitr::opts_chunk$set(
  dev = "cairo_pdf",
  fig.width = 5,
  fig.height = 5,
  fig.align = 'center',
  echo = FALSE,
  message = FALSE,
  warning = FALSE,
  error = FALSE,
  results = 'markup'
)

# Load required libraries
library("tidyverse")
library("patchwork")
library("glue")
library("scales", warn.conflicts = FALSE)
library("extrafont")
library("tinytex")
library("knitr")
library("tidyr")
library("latex2exp")
library("MASS")
library("kableExtra")

theme_set(theme_minimal(base_family = "Roboto Condensed"))

conflicted::conflicts_prefer(
  readr::col_factor(),
  purrr::discard(),
  dplyr::lag(),
  readr::parse_date(),
  kableExtra::group_rows(),
  dplyr::select
)
library(ggplot2)
library(dplyr)

# Define grids
lambda_grid <- tibble(lambda = seq(0, 5, length.out = 501), ll = -lambda)

eta_grid <- tibble(eta = seq(-10, 2, length.out = 501), ll_eta = -exp(eta))

# Plot 1: lambda space
p1 <- ggplot(lambda_grid, aes(lambda, ll)) +
```

```
  geom_line() +
  geom_vline(xintercept = 0, linetype = "dashed") +
  annotate(
    "text",
    x = 1.8,
    y = -1,
    label = "MLE at \\hat{\\lambda}=0",
    hjust = 0
  ) +
  labs(
    title = "Poisson log-likelihood when x = 0 (lambda space)",
    x = expression(lambda),
    y = "log-likelihood (up to constant)"
  ) +
  theme_minimal()

# Plot 2: eta space
p2 <- ggplot(eta_grid, aes(eta, ll_eta)) +
  geom_line() +
  annotate(
    "text",
    x = -6,
    y = -0.002,
    label = "pushed to -infinity as lambda→0",
    hjust = 0
  ) +
  labs(
    title = "Poisson log-likelihood when x = 0 (eta = log lambda space)",
    x = expression(eta),
    y = "log-likelihood (up to constant)"
  ) +
  theme_minimal()

p1 / p2
```