# STA 6351, Report.1.1

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### 1.1

**A life-testing experiment is conducted on $n$ identical components. Each component's lifetime is modeled as independent and exponentially distributed with mean $\theta > 0$. The test is terminated when $r$-th failure occurs, where $1 \leq r \leq n$ is fixed in advance (Type II censoring). Let the vector of $n$ failure times be $\mathbf{t}_n$. The ordered failure times are**

$$t_{(1)} < t_{(2)} < \cdots < t_{(r)},$$

**and denote the censoring time by $t_{(r)}$.**

**Derive the likelihood function $L(\theta|\mathbf{t})$ for the Type II censored sample. Be explicit about how the contribution of the censored observations is incorporated, and show that your likelihood reduces to the complete-data likelihood when $r = n$.**

Suppose $T_1, \ldots, T_n \overset{iid}{\sim} \text{Exp}(\theta)$ with mean $\theta > 0$ (rate $\lambda = 1/\theta$). Under Type II censoring, we observe thes first $r$ ordered failure times

$$t_{(1)} < \cdots < t_{(r)},$$

and the remaining $n - r$ units are right-censored at $t_{(r)}$. For any iid model with density $f(\cdot \mid \theta)$ and survival function $S(\cdot \mid \theta)$, the likelihood under Type II censoring is

$$L(\theta \mid \mathbf{t}) \;\propto\; \left\{ \prod_{i=1}^{r} f\big(t_{(i)} \mid \theta\big) \right\} \left\{ S\big(t_{(r)} \mid \theta\big) \right\}^{n-r}.$$

For the exponential distribution with rate $\lambda = 1/\theta$,

$$f(t \mid \theta) = \lambda e^{-\lambda t}, \qquad S(t \mid \theta) = e^{-\lambda t}.$$

Therefore,

$$L(\theta \mid \mathbf{t}) \;\propto\; \left( \prod_{i=1}^{r} \lambda e^{-\lambda t_{(i)}} \right) \left( e^{-\lambda t_{(r)}} \right)^{n-r}.$$

Simplifying,

$$L(\theta \mid \mathbf{t}) = \lambda^r \exp\left( -\lambda \left( \sum_{i=1}^{r} t_{(i)} + (n-r) t_{(r)} \right) \right).$$

In terms of $\theta$ (with $\lambda = 1/\theta$),

$$L(\theta \mid \mathbf{t}) \;\propto\; \theta^{-r} \exp\left( -\frac{1}{\theta} \Big( \sum_{i=1}^{r} t_{(i)} + (n-r)\, t_{(r)} \Big) \right)$$

**Check:** If $r = n$, then $n - r = 0$ and $\sum_{i=1}^{n} t_{(i)} = \sum_{i=1}^{n} t_i$.
Thus

$$L(\theta \mid \mathbf{t}) \;\propto\; \theta^{-n} \exp\left(-\frac{1}{\theta} \sum_{i=1}^{n} t_i\right),$$

which is exactly the complete-data likelihood for $n$ exponential observations.

We see here that all of the censored observations are incorporated into the likelihood function via the survival function.

**Obtain the log-likelihood, $\ell(\theta|\mathbf{t}_n)$.**

The log-likelihood of $\theta$ would be written as,

$$\ell(\theta \mid \mathbf{t}_n) = -\,r \log \theta \;-\; \frac{1}{\theta} \left( \sum_{i=1}^{r} t_{(i)} \;+\; (n-r)\, t_{(r)} \right).$$